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ON THE APPROXIMATE SOLUTION OF THE J. BALL'S BEAM EQUATION IN THE CASE OF PRESSURE DEPENDENCE OF EFFECTIVE VISCOSITY

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Abstract. An initial boundary value problem is posed for the J. Ball integro-differential equation, which describes the dynamic state of a beam. The solution is approximated utilizing the Galerkin method, stable symmetrical difference scheme and the Jacobi iteration method. This paper presents the approximate solution to one practical problem, particularly, the results of numerical computations of the initial boundary value problem for an iron beam. In the present article the case where the effective viscosity depends on the pressure is discussed. The results of numerical calculations qualitatively satisfactorily describe the process under consideration.

Keywords and phrases: Nonlinear dynamic beam equation, J. Ball equation, Galerkin method, implicit symmetric difference scheme, Jacobi iterative method, iron beam, numerical realization.

AMS subject classification (2010): 65M60, 65M06, 65Q10, 65M15.

1 Statement of the problem. Let us consider J. Ball's beam nonlinear integro-differential equation (see [1])

$$u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t)$$
$$-\left(\beta+\kappa\int_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t)-\sigma\left(\int_0^L u_x(x,t)\,u_{xt}(x,t)\,dx\right) \tag{1}$$
$$\times u_{xx}(x,t)=f(x,t),\qquad 0<x<L,\qquad 0<t\le T,$$

with the initial boundary conditions

$$u(x,0)=u^0(x),\qquad u_t(x,0)=u^1(x), \tag{2}$$

$$u(0,t)=u(L,t)=0,\qquad u_{xx}(0,t)=u_{xx}(L,t)=0. \tag{3}$$

Here $\alpha,\gamma,\kappa,\sigma,\beta$, and δ are given constants, among which the first four are positive numbers, while $u^0(x)\in W_2^2(0,L)$ and $u^1(x)\in L_2(0,L)$, are given functions such that $u^0(0)=u^1(0)=u^0(L)=u^1(L)=0$.

The right hand side function $f(x,t)\in L_2((0,L)\times(0,T))$. We suppose that there exits a solution $u(x,t)\in W_2^2((0,L)\times(0,T))$ of the problem (1)-(3).

The present article is a direct continuation of the articles [2]-[6] that considered an initial boundary value problem for the J. Ball integro-differential equation, which describes the dynamic state of a beam. The solution is approximated utilizing the Galerkin method, stable symmetrical difference scheme and the Jacobi iteration method(see [7]). In the articles [2]-[3] the algorithm has been approved by tests. In the article [4]-[6] and the present paper the approximate solution to one practical problem, particularly, the results of numerical computations of the initial boundary value problem for an iron beam is considered. In the present article the case where the effective viscosity depends on the pressure is discussed. The results of numerical calculations qualitatively and satisfactorily describe the process under consideration. A physical model that J. Ball uses in the article [1] is taken from the handbook of Engineering Mechanics written by E. Mettler (see [8]). For this model he wrote the corresponding initial boundary value problem for the integro-differential equation of a beam (1). Here $\alpha, \gamma, \kappa, \sigma, \beta$ and δ are given constants from which the following five have the form

$$\alpha = \frac{E \cdot I}{\rho}, \quad \beta = \frac{E \cdot A \cdot \Delta}{L \cdot \rho}, \quad \gamma = \frac{\eta \cdot I}{\rho}, \quad \kappa = \frac{E \cdot A}{2L \cdot \rho}, \quad \sigma = \frac{A\eta}{L \cdot \rho}.$$

where E is the Young' modulus, A is the cross-section area, η is the effective viscosity, I is the cross-sectional second moment of area, ρ is the mass per unit length the reference configuration, L is beam length, Δ is beam length change (extension) and δ is the coefficient of external damping.

2 The numerical realization. For the approximate solution of an initial boundary value problem (1)-(3) several programs are composed in "Maple", several numerical experiments are carried out. This paper presents the approximate solution to the one practical problem, particularly, the results of numerical computations of the initial boundary value problem for an iron beam are represented in the tables.

Issues of the initial boundary value problem of the iron beam are studied for the following meanings of parameters: spatial, temporal, mathematical algorithm and physical nature of the beam. $L = 1\,\text{m}$, $T = 1\,\text{sec}$, the grid length of a spatial variable $H = 5$, the grid length of a temporal variable $M = 5$, the amount of coordinate functions in the Galerkin method $n = 5$; number of iterations $n_{iter} = 5$, $E = 1.9 \times 10^6\,\dfrac{\text{kg}}{\text{cm}^2}$, $\rho = 7.874\,\dfrac{\text{g}}{\text{cm}^3}$, $\Delta = 0.01\,\text{m}$, $A = 0.01\,\text{m}^2$, $I = 1000\,\text{Pa}$.

In this paper, we take the effective viscosity as a function of pressure and time :

$\eta(t) = (1000 + 100t)^{-1}$ $\alpha = 0.24613 \times 10^6 \cdot datvI$, $\beta = 241.3$, $\gamma = 0.12954 \times datv \cdot \eta$, $\kappa = 12065$, $\sigma = 0.0127 \times \eta$, and $\delta = 0$. The initial functions $u^0(x) = \sin\left(\dfrac{\pi x}{L}\right)$, $u^1(x) = 0$, the right-hand function $f(x,t) \equiv 0$. For each counting problem, we will see two options:

a) Simple model - for each specific t, we calculate η and take the coefficients in the corresponding differential equations to be constant across all time scales;

b) Complex model - in the differential equations, we take the coefficients as time-dependent at t for all time scales; we compare the results of the cases considered for the simple model with the results obtained with the complex model for different values of the spatial and temporal variables.

In the case of pressure dependence, the numerical results agreed with high accuracy.

In the presented paper the numerical computations of bending function of the beam $u(x,t)$ is presented for the several meanings of the following effective viscosity η:

Case 1- A simple model,
Case 2 - complex model. See Table 1;

$t \ \backslash \ x$	Case	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
$t = 0$	1	0	0.587785	0.951057	0.951057	0.587785	0
$t = 0$	2	0	0.587785	0.951057	0.951057	0.587785	0
$t = 0.1$	1	0	1.763356	2.853169	2.853169	1.763356	0
$t = 0.1$	2	0	1.763356	2.853169	2.853169	1.763356	0
$t = 0.2$	1	0	4.114494	6.657392	6.657392	4.114494	0
$t = 0.2$	2	0	4.114494	6.657392	6.657392	4.114494	0
$t = 0.3$	1	0	6.465628	10.461607	10.461607	6.465628	0
$t = 0.3$	2	0	6.465628	10.461607	10.461607	6.465628	0
$t = 0.4$	1	0	8.816755	14.265810	14.265810	8.816755	0
$t = 0.4$	2	0	8.816755	14.265810	14.265810	8.816755	0
$t = 0.5$	1	0	11.167873	18.069998	18.069998	11.167873	0
$t = 0.5$	2	0	11.167873	18.069998	18.069998	11.167873	0
$t = 0.6$	1	0	13.518978	21.874166	21.874166	13.518978	0
$t = 0.6$	2	0	13.518978	21.874166	21.874166	13.518978	0
$t = 0.7$	1	0	15.870069	25.678311	25.678311	15.870069	0
$t = 0.7$	2	0	15.870069	25.678311	25.678311	15.870069	0
$t = 0.8$	1	0	18.221143	29.482428	29.482428	18.221143	0
$t = 0.8$	2	0	18.221143	29.482428	29.482428	18.221143	0
$t = 0.9$	1	0	20.572197	33.286514	33.286514	20.572197	0
$t = 0.9$	2	0	20.572197	33.286514	33.286514	20.572197	0
$t = 1$	1	0	22.923229	37.090563	37.090563	22.923229	0
$t = 1$	2	0	22.923229	37.090563	37.090563	22.923229	0

Table 1.

Conclusion. As numerical experiments demonstrate, once the coefficients of effective viscosity η increase, concomitantly, the numerical values of displacement functions (curvature) $u(x,t)$ decrease for specific values of x and t. However, for every specific values of η, the numerical values of the displacement functions for specific x increase with the increase of time t. The numerical values of the displacement functions with respect to temporal variable t are symmetrical to the midpoint of beam at $x = L2$.

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