Proceedings of I. Vekua Institute
of Applied Mathematics
Vol. 75, 2025

ON THE DECOMPOSITION OF THE RATIONAL MATRIX FUNCTION

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Abstract. In this paper, we consider a decomposition method for a rational matrix function by using elementary actions on the rows and columns of this matrix. A rational matrix function is associated with the splitting type of a vector bundle, and because of this, this decomposition method can possibly be used for calculations in this case.

Keywords and phrases: Rational matrix function, vector bundle.

AMS subject classification (2020): 74F10, 74H10.

1 Description of the decomposition method using elementary actions

We define the identity matrix as I. Consider a special matrix, $E_{ij}(z)$, which contains zeros everywhere except at the intersection of the i-th row and the j-th column, provided $i \neq j$, where it features the polynomial $p_{ij}(z)$. If we then define a new matrix $E(z)$ as the sum of the identity matrix and this special matrix, $E(z) = I + E_{ij}(z)$, the determinant of $E(z)$ is by definition always 1 ($\det(E(z)) = 1$). Next, we introduce the matrix

$$M = \begin{pmatrix} e_{i_1} \\ e_{i_2} \\ \vdots \\ e_{i_p} \end{pmatrix}. \tag{1}$$

This matrix is composed of the row vectors e_i, where each vector e_i has a ± 1 in the i-th position and zeros elsewhere. The indices $i_1, i_2, \ldots, i_p$ ar the permutation of the numbers $1, 2, \ldots, p$. The sign ± 1 is specifically chosen to ensure that the determinant of M is exactly 1. The same logic applies if M is defined by its column vectors instead. Matrices that are either of type $E(z)$ or type M will henceforth be referred to as “elementary matrices”.
Consider the matrix

$$T = (t_{ij}) = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_p \end{pmatrix}, \tag{2}$$

where $t_i = (t_{i1},\ t_{i2},\ \ldots,\ t_{ip})$ for $i \in \{1,\ \ldots,\ p\}$. The action $E(z)T$ changes the i-th row of matrix T to $t_i + p_{ij}(z)t_j$, while all other rows remain unchanged. The action MT permutes the rows of matrix T. Similar actions apply when T is represented by its columns, i. e., when

$$T = (t_{ij}) = (t_1\ t_2 \ldots t_p), \tag{3}$$

where

$$t_j = \begin{pmatrix} t_{1j} \\ t_{2j} \\ \vdots \\ t_{pj} \end{pmatrix}, \tag{4}$$

for $j\ \in \{1,\ \ldots,\ p\}$. The action $TE(z)$ changes the j-th column of matrix T to $t_j + p_{ij}(z)t_i$, while all other columns remain unchanged. The action TM permutes the columns of matrix T. We call the multiplication of matrix T by elementary matrices $E(z)$ or M from the right or from the left an "elementary action" on T. Let P be a product of a finite number of $E(z)$ and M matrices. By definition, $\det(P) = 1$. Let's consider the following matrix $T(z)$

$$\sum_{k=-n}^{m} \frac{A_k}{z^k} = \frac{A_{-n}}{z^n} + \ldots + \frac{A_{-1}}{z} + A_0 + A_1 z + \ldots + A_m z^m, \tag{5}$$

where A_k are constant complex matrices of dimension $p{\times}p$ for $k{\in}\{-n, \ldots, m\}$, and $p,\ m,\ n\ \in\ \mathbb{N}$.

Theorem 1. *If $T(z) = \sum_{k=-n}^{m} \frac{A_k}{z^k}$ and det$(T(z)) \neq 0$, then a left decomposiotion of T(z) is possible through elementary actions*

$$T(z) = P_+(z) z^{diag(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p)} Q_-(z), \tag{6}$$

where

1. *$P_+(z)$ is a polynomial matrix function such that det$(P_+(z)) = 1$;*
2. *$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p$ are the integer numbers;*
3. *$Q_-(z)$ is a polynomial matrix function of z^{-1}, with det$(Q_-(\infty)) \neq 0$.*

Proof. If $P(z)$ is a polynomial matrix function with $\det(P(z)) = 1$, then $P^{-1}(z)$ is also a polynomial matrix function with $\det(P^{-1}(z)) = 1$. Each element of the matrix $T(z)$ consists of a finite number of terms. We can formally assume that each element of the matrix $T(z)$ has the same number of terms (the coefficients of some terms might be zero, but we still account for them in our discussion).

Through the action $t_i(z) + p_{ij}(z)t_j(z)$, it is possible to cancel the highest degree of z in the row vector $t_i(z)$ and, consequently, reduce the highest degree of z. This operation reduces the number of terms in the row vector $t_i(z)$ and does not form any new terms. We can begin the process of reducing the highest exponent of z in the matrix $T(z)$ by multiplying $T(z)$ on the left by matrices of type $E(z)$, which corresponds to the action $t_i(z)+p_{ij}(z)t_j(z)$. Since the number of terms in the row vectors $t_i(z)$ and $t_j(z)$ $(i \neq j)$ is finite and no new terms are formed by this action, the process of decreasing the highest degree of z in the rows of $T(z)$ cannot continue infinitely. This means that after a finite number of operations, we will obtain a matrix $P_1(z)T(z)$. In this new matrix, for a row $t_i(z)$, either it contains no more powers of z, i. e., $t_i(z) = (0,\ 0,\ \ldots,\ 0)$, or the highest degree of z in the row $t_i(z)$ no longer reduces. Since $\det(T(z)) = \det(P_1(z)T(z))$ and $\det(T(z)) \neq 0$, the case $t_i(z) = (0,\ 0,\ \ldots,\ 0)$ cannot occur. This implies that the highest degree of z in the row $t_i(z)$ no longer reduces. Let's call this degree α_1. We can assume that the power α_1 of z, is located in the first column of the first row of the matrix $P_1(z)T(z)$. If it is not, we can achieve this by multiplying the matrix T(z) from the left by an M-type matrix, and then from the right. α_1 is the smallest power of z for the matrix $T(z)$ that cannot be reduced further by elementary actions on its rows, then

$$P_1(z)T(z) = \begin{pmatrix} \widetilde{t}_1(z) \\ \widetilde{t}_2(z) \\ \vdots \\ \widetilde{t}_p(z) \end{pmatrix} = \begin{pmatrix} z^{\alpha_1}q_{11}(z) & z^{\alpha_1}q_{12}(z) & \ldots & z^{\alpha_1}q_{1p}(z) \\ \widetilde{t}_{21}(z) & \widetilde{t}_{22}(z) & \ldots & \widetilde{t}_{2p}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{t}_{p1}(z) & \widetilde{t}_{p2}(z) & \ldots & \widetilde{t}_{pp}(z) \end{pmatrix}, \tag{7}$$

where $P_1(z)$ is a product of a finite number of $E(z)$ and M matrices. The highest degree of z is less than α_1 in each member of the first column in the matrix $P_1(z)T(z)$. The vector function $(q_{11}(z),\ q_{12}(z),\ \ldots,\ q_{1p}(z))$ is a polynomial in z^{-1}, and $(q_{11}(\infty),\ q_{12}(\infty),\ \ldots,\ q_{1p}(\infty)) = (c_1,\ *,\ \ldots,\ *)$, with $c_1 \neq 0$.

We continue the elementary actions for the remaining rows with indices from 2 to p. After a finite number of operations, we will find a row where the highest power of z no longer reduces. Let's call this degree α_2. By the definition of α_1, we have $\alpha_1 \leq \alpha_2$. In this case, we can also assume that the power α_2 of z, is located in the second column of the second row of the matrix $P_2(z)P_1(z)T(z)$. α_2 is the smallest power of z for the matrix $T(z)$ that cannot be reduced further by elementary actions on the rows with indices from 2 to p, and

$$P_2(z)P_1(z)T(z) = \begin{pmatrix} z^{\alpha_1}q_{11}(z) & z^{\alpha_1}q_{12}(z) & \ldots & z^{\alpha_1}q_{1p}(z) \\ r_{21}(z) & z^{\alpha_2}r_{22}(z) & \ldots & z^{\alpha_2}r_{2p}(z) \\ \vdots & \vdots & \ddots & \vdots \\ s_{p1}(z) & s_{p2}(z) & \ldots & s_{pp}(z) \end{pmatrix}, \tag{8}$$

the highest degree of z is less than α_1 in each member of the first column of $P_2(z)P_1(z)T(z)$, and the highest degree of z is less than α_2 in each member of the second column. $P_2(z)P_1(z)$ is a product of a finite number of $E(z)$ and M matrices. The vector function $\left(\frac{1}{z^{\alpha_2}}r_{21}(z),\ r_{22}(z),\ \ldots,\ r_{2p}(z)\right)$ is a polynomial in z^{-1}, and $\left(\frac{1}{z^{\alpha_2}}r_{21}(z),\ r_{22}(z),\ \ldots,\ r_{2p}(z)\right)|_{z=\infty} = (0,\ c_2,\ *,\ \ldots,\ *)$, with $c_2 \neq 0$. After a finite number of operations, we will get

$$P_p(z)\cdots P_2(z)P_1(z)T(z) = \begin{pmatrix} z^{\alpha_1}q_{11}(z) & z^{\alpha_1}q_{12}(z) & \dots & z^{\alpha_1}q_{1p}(z) \\ r_{21}(z) & z^{\alpha_2}r_{22}(z) & \dots & z^{\alpha_2}r_{2p}(z) \\ \vdots & \vdots & \ddots & \vdots \\ h_{p1}(z) & h_{p2}(z) & \dots & z^{\alpha_p}h_{pp}(z) \end{pmatrix}, \quad (9)$$

the integers are $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p$. $P_p(z)\cdots P_2(z)P_1(z)$ is a product of a finite number of $E(z)$ and M matrices, so $P_p(z)\cdots P_2(z)P_1(z) = P_+(z)$ is a polynomial matrix function with $\det(P_+(z)) = 1$. $P_+^{-1}(z)$ is also a polynomial matrix function with $\det(P_+^{-1}(z)) = 1$. The vector function $\left(\frac{1}{z^{\alpha_p}}h_{p1}(z),\ \frac{1}{z^{\alpha_p}}h_{p2}(z),\ \ldots,\ h_{pp}(z)\right)$ is a polynomial in z^{-1}, where $\left(\frac{1}{z^{\alpha_p}}h_{p1}(z),\ \frac{1}{z^{\alpha_p}}h_{p2}(z),\ \ldots\ h_{pp}(z)\right)|_{z=\infty} = (0,\ 0,\ 0,\ \ldots,\ 0,\ c_p)$, with $c_p \neq 0$, and

$$Q_-(z) = \begin{pmatrix} q_{11}(z) & q_{12}(z) & \dots & q_{1p}(z) \\ \frac{1}{z^{\alpha_2}}r_{21}(z) & r_{22}(z) & \dots & r_{2p}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{z^{\alpha_p}}h_{p1}(z) & \frac{1}{z^{\alpha_p}}h_{p2}(z) & \dots & h_{pp}(z) \end{pmatrix} \quad (10)$$

is a matrix function of z^{-1};

$$\det(Q_-(\infty)) = \begin{vmatrix} c_1 & * & \dots & * \\ 0 & c_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_p \end{vmatrix} = c_1c_2\cdots c_p \neq 0, \quad (11)$$

$$P_+(z)T(z) = \begin{pmatrix} z^{\alpha_1} & 0 & \dots & 0 \\ 0 & z^{\alpha_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z^{\alpha_p} \end{pmatrix} Q_-(z) = z^{\mathrm{diag}(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p)}Q_-(z). \quad (12)$$

Thus, $T(z) = P_+^{-1}(z)z^{\mathrm{diag}(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p)}Q_-(z)$. The matrices $P_+(z)$ and $Q_-(z)$ are not unique, but the integers $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p$ are unique and depend only on the matrix $T(z)$. Suppose there exists another representation of $T(z)$ with other integers $\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_p$, i. e.,

$$T(z) = P_{1+}(z)z^{\mathrm{diag}(\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_p)}Q_{1-}(z) = P_{2+}(z)z^{\mathrm{diag}(\gamma_1,\,\gamma_2,\,\ldots,\gamma_p)}Q_{2-}(z), \quad (13)$$

with $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p$ and $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_p$. Also, $\alpha_1 + \alpha_2 + \ldots + \alpha_p = \gamma_1 + \gamma_2 + \ldots + \gamma_p$. By definition, $\alpha_j \leq \gamma_j$, so we can write $\gamma_j = \alpha_j + \delta_j$ with $\delta_j \geq 0$ for $j \in \{1,\ \ldots,\ p\}$. Substituting this into the sum, we get $\gamma_1 + \ldots + \gamma_p = \alpha_1 + \ldots + \alpha_p + (\delta_1 + \ldots + \delta_p)$. From this equation, we have $\alpha_1 + \ldots + \alpha_p = \alpha_1 + \ldots + \alpha_p + (\delta_1 + \ldots + \delta_p)$, which implies $\delta_1 + \ldots + \delta_p = 0$. Since $\delta_j \geq 0$ for all $j \in \{1,\ \ldots,\ p\}$, it must be that $\delta_j = 0$ for all $j \in \{1,\ \ldots,\ p\}$. This proves that $\alpha_j = \gamma_j$ for all $j \in \{1,\ \ldots,\ p\}$, and therefore $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p$ are the unique left partial indices. The proof is analogous for the right factorization of the matrix T(z). □

Based on Theorem 1, it follows that: If $T(z) = \sum_{k=-n}^{m} \frac{A_k}{z^k}$ and $\det(T(z)) \neq 0$, then a right decomposiotion of T(z) is possible through elementary actions

$$T(z) = \widetilde{Q}_-(z) z^{\mathrm{diag}(\beta_1,\ \beta_2,\ \dots,\ \beta_p)} \widetilde{P}_+(z), \tag{14}$$

where
1. $\widetilde{P}_+(z)$ is a polynomial matrix function such that $\det(\widetilde{P}_+) = 1$;
2. $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_p$ are the integer numbers;
3. $\widetilde{Q}_-(z)$ is a polynomial matrix function of z^{-1}, with $\det(\widetilde{Q}_-(\infty)) \neq 0$.

2 The decomposition of a rational matrix function

Now, let's apply the method described above to rational matrix functions.

Theorem 2. *If $T(z)$ is a rational matrix function and det$(T(z)) \neq 0$, then a left decomposiotion of T(z) is possible through elementary actions*

$$T(z) = R_+(z) z^{diag(\alpha_1,\ \alpha_2,\ \dots,\ \alpha_p)} Q_-(z), \tag{15}$$

where
1. *$R_+(z)$ is an entire matrix function;*
2. *$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p$ are the integer numbers;*
3. *$Q_-(z)$ is a polynomial matrix function of z^{-1}, with det$(Q_-(\infty)) \neq 0$.*

Proof. $T(z) = \left(\frac{f_{ij}(z)}{g_{ij}(z)}\right)$, where $f_{ij}(z)$ and $g_{ij}(z)$ are polynomials in z. The denominator can be written as $g_{ij}(z) = c_{ij}(z - a_1^{ij})^{k_1^{ij}}(z - a_2^{ij})^{k_2^{ij}} \cdots (z - a_{ij}^{ij})^{k_{ij}^{ij}}$, where $c_{ij} \in \mathbb{C}$ and $c_{ij} \neq 0$, for $i, j \in \{1,\ \dots,\ p\}$. Thus, $T(z)$ has a finite number of singular points. These singular points are poles of order ≥ 1 and are located at the roots of the polynomials $g_{ij}(z)$.
Let the set of all singular points be $\{b_1,\ b_2,\ \dots,\ b_s\}$ with multiplicities $k_1,\ k_2,\ \dots,\ k_s$. We can give every element of the matrix $T(z)$ a common denominator, specifically the polynomial $c(z - b_1)^{k_1}(z - b_2)^{k_2} \cdots (z - b_s)^{k_s}$, where $c \in \mathbb{C}$ and $c \neq 0$. After this, the matrix $T(z)$ will have the following form

$$T(z) = \left(\frac{\widetilde{f}_{ij}(z)}{c(z - b_1)^{k_1}(z - b_2)^{k_2} \cdots (z - b_s)^{k_s}} \right), \tag{16}$$

where $\widetilde{f}_{ij}(z)$ are polynomials in z for $i, j \in \{1,\ \dots,\ p\}$. Let $q = k_1 + k_2 + \ldots + k_s$. Then $c(z - b_1)^{k_1}(z - b_2)^{k_2} \cdots (z - b_s)^{k_s} = cz^q + d_1 z^{q-1} + \ldots + d_{q-1} z + d_q$. This can be written as $z^q(c + d_1 z^{-1} + \ldots + d_{q-1} z^{1-q} + d_q z^{-q}) = z^q r(z)$, where $r(z) = c + d_1 z^{-1} + \ldots + d_{q-1} z^{1-q} + d_q z^{-q}$ is a scalar polynomial in z^{-1} and $r(\infty) = c \neq 0$. From $r(z) = \frac{cz^q + d_1 z^{q-1} + \ldots + d_{q-1} z + d_q}{z^q}$, it follows that $r^{-1}(z) = \frac{z^q}{cz^q + d_1 z^{q-1} + \ldots + d_{q-1} z + d_q}$, which gives $r^{-1}(0) = c^{-1} \neq 0$.

$T(z) = \left(\frac{\widetilde{f}_{ij}(z)}{z^q(c+d_1z^{-1}+...+d_{q-1}z^{1-q}+d_qz^{-q})}\right) = \left(\frac{\widetilde{f}_{ij}(z)}{z^q r(z)}\right) = \frac{1}{r(z)}\left(\frac{\widetilde{f}_{ij}(z)}{z^q}\right)$. This implies that $r(z)T(z) = \left(\frac{\widetilde{f}_{ij}(z)}{z^q}\right)$, so there exist $n, m \in \mathbb{N}$ such that $r(z)T(z) = \sum_{k=-n}^{m} \frac{A_k}{z^k}$. From Theorem 1, we have

$$\sum_{k=-n}^{m} \frac{A_k}{z^k} = P_+(z)z^{\mathrm{diag}(\alpha_1,\ \alpha_2,\ ...,\ \alpha_p)}Q_-(z).$$

Therefore $r(z)T(z) = P_+(z)z^{\mathrm{diag}(\alpha_1,\ \alpha_2,\ ...,\ \alpha_p)}Q_-(z)$, and we have

$$T(z) = (r^{-1}(z)P_+(z))z^{\mathrm{diag}(\alpha_1,\ \alpha_2,\ ...,\ \alpha_p)}Q_-(z). \tag{17}$$

$R_+(z) = r^{-1}(z)P_+(z)$ is an entire matrix function (moreover, a rational matrix function). $\det(R_+(z)) = (r^{-1}(z))^p\det(P_+(z)) = (r^{-1}(z))^p \neq 0$ and is holomorphically invertible in a neighborhood of the point 0. □

Based on Theorem 2, it follows that: If $T(z)$ is a rational matrix function and $\det(T(z)) \neq 0$, then a right decomposition of T(z) is possible through elementary actions

$$T(z) = \widetilde{Q}_-(z)z^{\mathrm{diag}(\beta_1,\ \beta_2,\ ...,\ \beta_p)}\widetilde{R}_+(z), \tag{18}$$

where

1. $\widetilde{R}_+(z)$ is an entire matrix function;
2. $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_p$ are the integer numbers;
3. $\widetilde{Q}_-(z)$ is a polynomial matrix function of z^{-1}, with $\det(\widetilde{Q}_-(\infty)) \neq 0$.

A possible application is as follows: The splitting type of a vector bundle induced by a Fuchsian system of differential equations is known to coincide with the partial indices of a specific rational matrix function (see [5], [6]). This function is constructed using a constant, nondegenerate matrix and a matrix of the form $(z-s)^{\mathrm{diag}(0,\ ...,\ 0,\ \pm1,\ ...,\ \pm1,\ 0,\ ...,\ 0)}$, where "s" denotes a singular point of the original Fuchsian system (see [7]).

These indices depend solely on the matrix function being factored. Consequently, the described method for decomposing a rational matrix function can be used to calculate its partial indices, which is equivalent to determining the splitting type of the vector bundle (see [5], [6]).

Acknowledgement. Research supported by Shota Rustaveli National Scientific Foundation of Georgia, Grant Agreement FR22-354.

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