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# A NOTE ON FRACTIONAL-TYPE GAUSSIAN FUNCTIONS

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*Dedicated to our dear friend Prof. Dr. George Jaiani on his 80th birthday*

**Abstract**. Using the reciprocal of fractional exponential functions, in terms of Bell polynomials, the fractional versions of the Gauss' and super-Gauss' functions, both in the classical and Laguerre-type case are presented.

**Keywords and phrases**: Fractional exponential function, Laguerre-type fractional exponential function, fractional Gauss' functions, fractional super-Gauss' functions.

**AMS subject classification (2020):** 26A33, 34A08, 12E10.

## 1 Introduction

The great interest in the fractional calculus [1, 2, 3] and expansions in fractional series of functions [4] has provided the motivation to extend various functions and elements of number theory to this field of interest. This topic falls under the study of generalized forms of polynomials and special numbers, as studied in the works of Booth and Hassen [5], Geleta and Hassen [6], Hu and Kim [7], Miloud and Tiachachat [8], and many others. Recently we have introduced fractional versions of the Bernoulli and Euler numbers [9], as well as their Laguerrian analogues [10].

In this same area, the introduction of fractional models of population dynamics [11] and extensions to the fractional case of the Laplace transform [12] has found its place.

In this paper, based on the construction of the reciprocal of fractional exponential functions, in terms of Bell polynomials, we aim to introduce fractional Gaussian functions and their Laguerrian analogues. Some graphs also show the cases of the super-Gaussian functions and the trend of the parameters of their distributions (second moment, variance and kurtosis).

The introduction of these fractional Gaussian and super-Gaussian distributions should allow greater flexibility in analyzing the behavior of natural phenomena, since the shapes real world often appear in fractional forms.

## 2 The fractional exponential function

For any real number $\alpha > 0$, the fractional derivative $D_x^\alpha$ [13], according the Euler definition, writes

$$D_x^\alpha x^n = \begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, x^{n-\alpha}\,, & \text{if} \quad n > \lceil\alpha\rceil - 1\,, \\ 0\,, \quad \text{if} \quad n = 0,1,\ldots,\lceil\alpha\rceil - 1\,, \end{cases} \tag{1}$$

where $n \geq 0$ and $\lceil\alpha\rceil$ denotes the ceiling function, that is the smallest integer greater than or equal to $\alpha$.
If $c$ is a constant then $D_x^\alpha c = 0$.

The fractional exponential function (depending on the parameter $\alpha$) is defined by

$$\mathrm{Exp}_\alpha(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \ldots . \tag{2}$$

It is an eigenfunction of the operator $D_x^\alpha$, since it results

$$D_x^\alpha \mathrm{Exp}_\alpha(xt) = t^\alpha\, \mathrm{Exp}_\alpha(xt)\,. \tag{3}$$

## 3 Fractional Gaussian functions

In a recent article (see [12] and the references therein) we have introduced the reciprocal of the fractional exponential function (2), using Bell's polynomials. Namely, the following equation holds

$$\begin{aligned} [\mathrm{Exp}_\alpha(t)]^{-1} &= 1 + \sum_{n=1}^{\infty}\sum_{h=1}^{n} (-1)^h\, \Gamma(h\alpha+1)\, B_{n,h}(1,1,\ldots,1)\, t^{n\alpha} \\ &= \sum_{n=0}^{\infty}\sum_{h=0}^{n} (-1)^h\, \Gamma(h\alpha+1)\, S_{n,h}\, \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\,, \end{aligned} \tag{4}$$

where $S_{n,h}$ denote Stirling numbers of the second kind, and $S_{0,0} := 1$.

Using the same technique, for any fixed fractional $\alpha$, we define the fractional Gauss's functions by letting

$$\begin{aligned} \mathrm{Exp}_\alpha(-t^2) = \quad & \sum_{n=0}^{\infty} \frac{t^{2n\alpha}}{n!} \sum_{k=0}^{n} (-1)^k\, k! \\ & \times B_{n,k}\left(\frac{1!}{[\Gamma(\alpha+1)]}, \ldots, \frac{(n-k+1)!}{[\Gamma((n-k+1)\alpha+1)]}\right) . \end{aligned} \tag{5}$$

Graphs of the resulting functions are depicted in Figures 1, 2, 3. Figure 2 shows that, assuming $\alpha < 1$. the convergence to the classical Gauss' function from below holds, while Figure 3 shows that the convergence from above holds when $\alpha > 1$.

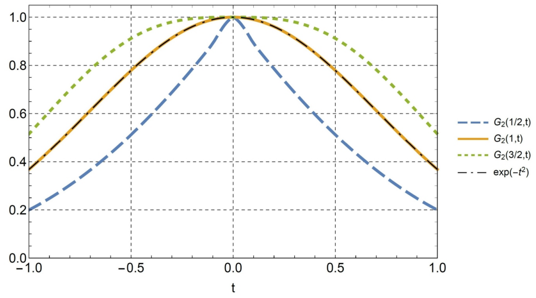


Figure 1: Graphs of particular fractional Gauss' functions.

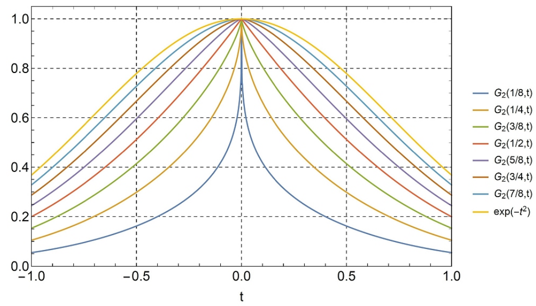


Figure 2: Graphs of the fractional Gauss' functions, for particular values of $\alpha < 1$.

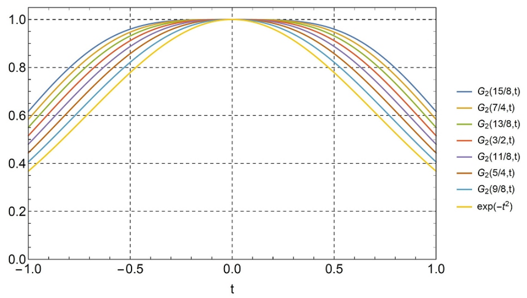


Figure 3: Graphs of the fractional Gauss' functions, for particular values of $\alpha > 1$.

# 4 Fractional Gaussian distributions

The normalizing constant of the relevant fractional Gaussian distribution is written

$$A_\alpha = \int_{-\infty}^{+\infty} \mathrm{Exp}_\alpha(-t^2)dt\,, \tag{6}$$

the mean value is

$$\mu_\alpha = (\mu_1)_\alpha = \frac{1}{A}\int_{-\infty}^{+\infty} t\ \mathrm{Exp}_\alpha(-t^2)dt = 0\,, \quad \forall\,\alpha, \tag{7}$$

and the second moment is written

$$(\mu_2)_\alpha = \frac{1}{A}\int_{-\infty}^{+\infty} t^2 \operatorname{Exp}_\alpha(-t^2)dt\,. \tag{8}$$

Then the relevant variance is given by

$$\sigma_\alpha^2 = (\mu_2)_\alpha - (\mu_2)_\alpha^2\,, \tag{9}$$

and the kurtosis by

$$\kappa_\alpha = \frac{(\mu_4)_\alpha}{\sigma_\alpha^4}\,. \tag{10}$$

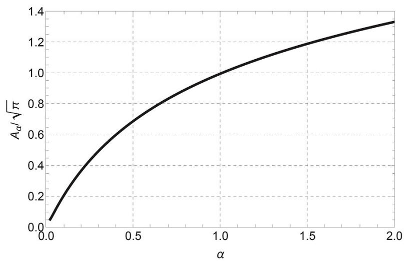


Figure 4: Normalizing constant of fractional Gaussian distributions, as functions of $\alpha$.

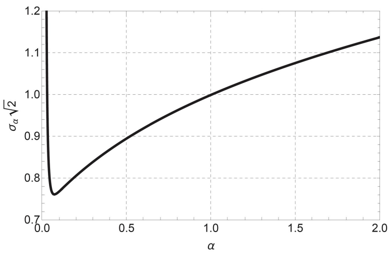


Figure 5: Variance of fractional Gaussian distributions, as functions of $\alpha$.

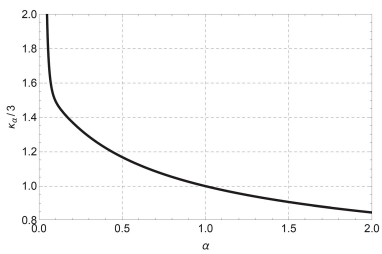


Figure 6: Kurtosis of fractional Gaussian distributions, as functions of $\alpha$.

The normalizing constant, the variance and the kurtosis of fractional Gaussian distributions, as functions of $\alpha$, are depicted in Figures 4, 5 and 6.

# 5 Fractional super-Gaussian functions

Super-Gaussian functions are important in several fields of probability, physics, engineering and even in medical sciences.

Graphs of super-Gauss' functions are depicted in Figures 7, 8, 9. Figure 8 shows that, assuming $\alpha < 1$. the convergence to the classical Gauss' function from below holds, while Figure 9 shows that the convergence from above holds when $\alpha > 1$.

$$\begin{aligned}\operatorname{Exp}(-t^4) = \quad & \sum_{n=0}^{\infty} \frac{t^{4n\alpha}}{n!} \sum_{k=0}^{n} (-1)^k\, k! \\ & \times B_{n,k}\left(\frac{1!}{[\Gamma(\alpha+1)]}, \ldots, \frac{(n-k+1)!}{[\Gamma((n-k+1)\alpha+1)]}\right). \quad (11)\end{aligned}$$

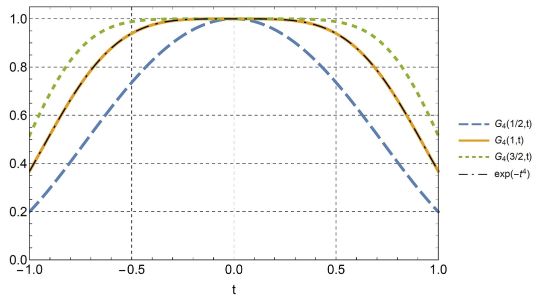


Figure 7: Graphs of particular fractional super-Gauss' functions.

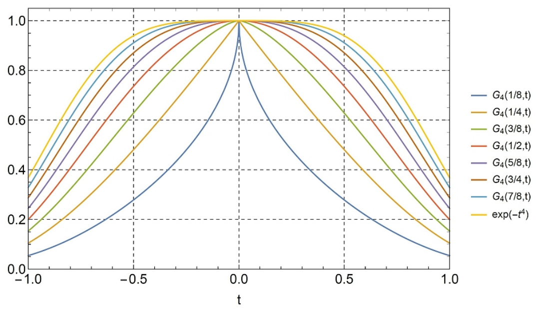


Figure 8: Graphs of the fractional super-Gauss' functions, for particular values of $\alpha < 1$.

# 6 Fractional super-Gaussian distributions

We show the main parameters of fractional super-Gaussian distributions, as functions of $\alpha$. Namely, their normalizing constants, variance and kurtosis are represented in Figures 10, 11 and 12.

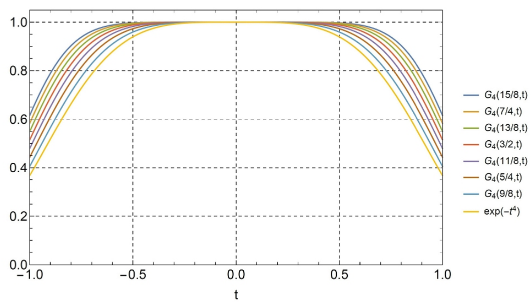


Figure 9: Graphs of the fractional super-Gauss' functions, for particular values of $\alpha > 1$.

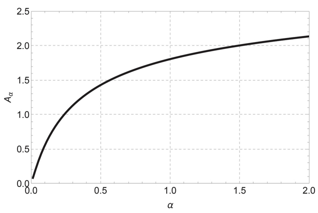


Figure 10: Normalizing constant of fractional super-Gaussian distributions, as functions of $\alpha$.

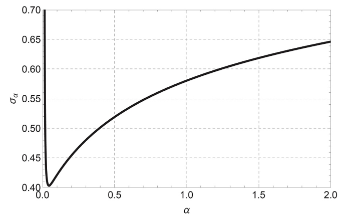


Figure 11: Variance of fractional super-Gaussian distributions, as functions of $\alpha$.

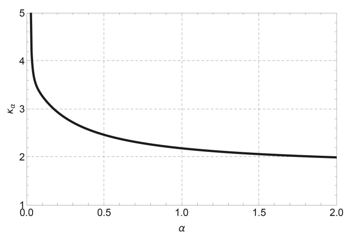


Figure 12: Kurtosis of fractional super-Gaussian distributions, as functions of $\alpha$.

# 7 The Laguerre-type case

An analog of the ordinary derivative is the so-called Laguerre derivative, as reported in [14, 15].

The first order Laguerre derivative, defined as $D_L := D_x x D_x = D_x + x\,D_x^2$ introduces a linear differential isomorphism into the space of analytic functions of the variable $x$, such that the differentiation properties are preserved. An eigenfunction of this operator is the first order Laguerre-type exponential

$$e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}\,, \tag{12}$$

since it results $\forall a \in \mathbf{C}$

$$D_L e_1(ax) = a e_1(ax)\,, \tag{13}$$

The $n$th order Laguerre derivative is defined as

$$\begin{aligned} D_{nL} := Dx \cdots DxDxD = D\left(xD + x^2D^2 + \cdots + x^nD^n\right) = \\ = S(n+1,1)D + S(n+1,2)xD^2 + \cdots + S(n+1,n+1)x^nD^{n+1}\,, \end{aligned} \tag{14}$$

where $S(n,k)$ denote Stirling numbers of the second kind.
The corresponding eigenfunctions is written

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}\,, \tag{15}$$

since it results $\forall a \in \mathbf{C}$

$$D_{nL}\; e_n(ax) = a e_n(ax)\,. \tag{16}$$

Actually the operator $D_L = DxD$ and its iterates

$$D_{nL} = DxDxDx \cdots DxD$$

are particular cases of the hyper-Bessel differential operators previously considered by Ditkin and Prudnikov [16]), and previously introduced, using differen names, by Dimovski [17], Delerue [18] and Kiryalova [3].

We have found the following results for the fractional Laguerre-type Gaussian and super-Gaussian distributions, which are similas to the ordinary ones. The relevant graphs are reported in Figures 13, 14, 15 and 16 below.

## 8 Conclusion

We have shown that the use of the fractional exponential function and the construction of the reciprocal of a fractional powers series by an application of Bell's polynomials allows the introduction of generalized forms of Gaussian and super-Gaussian functions, even in the Laguerrian case. These functions provide a wider possibility for modeling probabilistic distributions

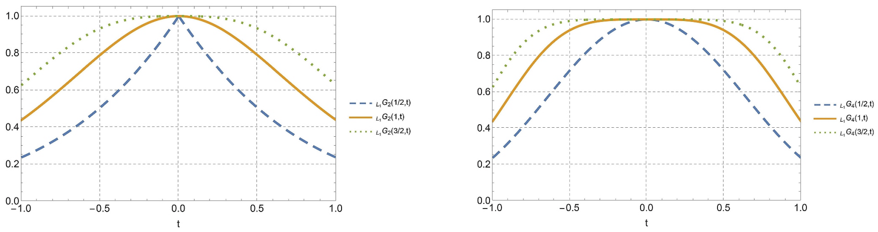


Figure 13: Graphs of Laguerre-type fractional functions vs the Super Laguerre-type.

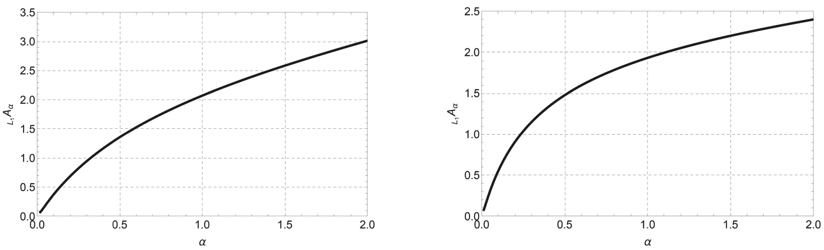


Figure 14: Normalizing constants of fractional Laguerre-type Gaussian distributions vs the super-Gaussian ones, as functions of $\alpha$.

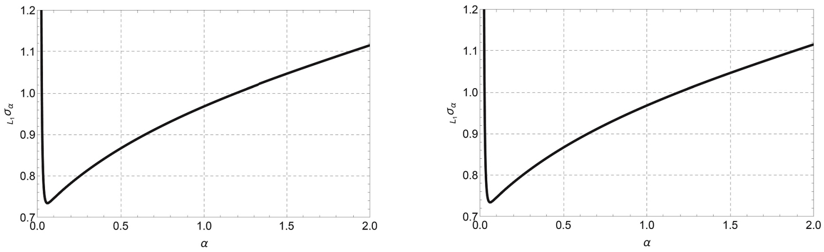


Figure 15: Variance of fractional Laguerre-type Gaussian distributions vs the super-Gaussian ones, as functions of $\alpha$.

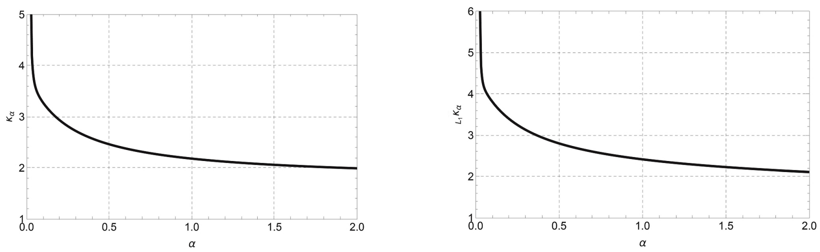


Figure 16: Kurtosis of fractional Laguerre-type Gaussian distributions vs the super-Gaussian ones, as functions of $\alpha$.

that can be used in applications, since the real world sometimes occurs in fractional forms. We have shown in graphs, dependent on the parameter $\alpha$

the trend of the statistical parameters that characterize the considered distributions. The leptokurtic trend of fractional Gaussian distributions when $\alpha < 1$ and the corresponding platykurtic trend of fractional super-Gaussian distributions were thus confirmed, in both the classical and Laguerre-type cases.

## References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. Theory and Applications of Fractional Differential Equations. *North-Holland Mathematical Studies. Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York*, 204:2006

[2] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin. Mittag-Leffler Functions. *Related Topics and Applications Springer, New York*, 2014, 443 pages, ISBN 978-3-662-43929-6

[3] V. Kiryakova. Generalized Fractional Calculus and Applications. *Pitman Res. Notes in Math. Ser., Longman, Harlow, U.K.*, 301:1994

[4] G. Groza, M. Jianu. Functions represented into fractional Taylor series. *ITM Web of Conferences* 29:2019, 01017 (ICCMAE 2018);
doi.org/10.1051/itmconf/20192901017

[5] R. Booth,, A. Hassen. Hypergeometric Bernoulli polynomials. *J. Algebra Number Theory, Academia*, 2(1):1-7, 2011

[6] H.L. Geleta, A. Hassen. Fractional hypergeometric zeta functions, *Ramanujan J.*, 41:421-436, 2016; doi.org/10.1007/s11139-015-9717-5

[7] S. Hu, M-S. Kim. On hypergeometric Bernoulli numbers and polynomials. *Acta Math. Hungar.*, 154:134-146, 2018

[8] M. Miloud, M. Tiachachat A new class of the r-Stirling numbers and the generalized Bernoulli polynomials, *Integers*, 20:1-8, 2020

[9] D. Caratelli, P. Natalini, P.E. Ricci. Fractional Bernoulli and Euler Numbers and related fractional polynomials. *A Symmetry in Number Theory, Symmetry*, 15(10):2023, 1900, doi.org/10.3390/sym15101900

[10] Ricci, P.E., Srivastava, R., Caratelli, D., Laguerre-type Bernoulli and Euler Numbers and related fractional polynomials, Mathematics, 12(3):2024, 381, doi.org/10.3390/math12030381

[11] D. Caratelli, P.E. Ricci. A note on fractional-type models of population dynamics. *Mathematical Modelling and Analysis*, 28(1):1-14, 2023

[12] D. Caratelli, P.E. Ricci. A Note on fractional exponential functions and applications, 2024 (submitted)

[13] L. Beghin, M. Caputo. Commutative and associative properties of the Caputo fractional derivative and its generalizing convolution operator.
doi.org/10.1016/j.cnsns.2020.105338

[14] G. Dattoli, P.E. Ricci. Laguerre-type exponentials, and the relevant L-circular and L-hyperbolic functions. *Georgian Math. J.*, 10:481-494, 2003

[15] P.E. Ricci, I. Tavkhelidze. An introduction to operational techniques and special polynomials. *Journal of Mathematical Sciences*, 157(1):2009

[16] A.P. Ditkin, V.A. Prudnikov. Integral Transforms and Operational Calculus. *Pergamon Press, Oxford, U.K.*, 1965

[17] I. Dimovski. Operational calculus for a class of differential operators. *C. R. Acad. Bulgare Sci.*, 19:1111-1114, 1966

[18] P. Delerue. Sur le calcul symbolique à $n$ variables et fonctions hyper-besséliennes (II). *Ann. Soc. Sci. Bruxelles*, 13, 229-274, 1953

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