Proceedings of I. Vekua Institute of Applied Mathematics
Vol. 74, 2024

DEFORMATION OF COMPLEX STRUCTURES, PARTIAL INDICES OF MATRIX FUNCTIONS

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Abstract. In this paper, we present the program implementation of the algorithm for computing the splitting type of holomorphic vector bundles induced from a Fuchsian system on the Riemann sphere. We develop a general approach to the problem by employing the methods of deformation of complex structures in the sense of Kodaira and Spencer. As a consequence, we calculate the dimension of the deformation space of complex structures of canonical deformation.

Keywords and phrases: Vector bundles, canonical extension, deformation, monodromy, Riemann surface.

AMS subject classification (2010): 15A23, 47A68, 34M03, 34M45, 30F20, 30F30.

Introduction

The topics addressed in this paper are closely related to the classification of holomorphic vector bundles on Riemann surfaces and have a topological character. On the other hand, the factorization problem of matrix functions has its own independent research methods that are more analytical in nature than topological. We utilized the progress achieved in the factoring technique of matrix functions for the investigation of invariants of holomorphic vector bundles induced from systems of differential equations.

Construction of the holomorphic vector bundle on Riemann surface by regular system of differential equations was considered by H. Röhrl [23] for the investigation of global behavior of the solutions of a system of differential equations. In particular, Röhrl used such a technique for the solution of Hilbert's 21 problem for the class of regular systems. The far-reaching generalization of Röhrl's approach to higher-dimensional complex manifolds were provided by P. Deligne [7]. Deligne considered the extension of the holomorphic vector bundle with meromorphic connection along the singular divisor and in such a way he obtained the nontrivial vector bundle on a complex manifold equipped with a logarithmic connection. In the last decade, several significant problems were solved by Deligne's approach to the global theory of differential equations, including Hilbert's famous 21st problem and its generalization [18], [25], [5], [13], [12], [11].

Here, we utilize Deligne's approach for regular systems of differential equations and construct a holomorphic vector bundle on the Riemann sphere from the Fuchsian system. We compute the splitting type of the canonical vector bundle and providing an algorithm for computing the dimension of the deformation space of holomorphic structures of the canonical bundle.

All the main results have program implementation in python 3.6+ and the numerical and symbolic computations were done using the mpmath library. The corresponding code is available at:

https://github.com/giorgi94/fuchsian--toolkit.

1. The canonical bundle induced from the Fuchsian system

Consider the non-resonant Fuchsian system of differential equations on the Riemann sphere $\mathbb{CP}^1$

$$df = \left(\sum_{j=1}^{m} \frac{A_j}{z - s_j} dz \right) f, \tag{1}$$

with poles at the points $s_1, ..., s_m$, $s_j \neq \infty$, where A_j, $j = 1, ..., m$ are constant matrices and $\sum_{j=1}^{m} A_j = 0$.

Denote by $S = \{s_1, ..., s_m\}$ the set of singular points of the system (1). Let $\gamma_1, ..., \gamma_m$ be closed paths with starting and end point z_0, $z_0 \overline{\in} S$ and going around $s_1, ..., s_m$. Let M_j be the monodromy matrices corresponding to s_j, $j = 1, ..., m$. Nondegenerate matrices M_j depend only on homotopy class $[\gamma_j]$ of γ_j and therefore defines the monodromy representation

$$\rho : (\mathbb{CP}^1 \setminus \{s_1, ..., s_m\}, z_0) \to GL_n(\mathbb{C}), \quad \rho([\gamma_j]) = M_j, \quad j = 1, ..., m. \tag{2}$$

Let $E_j = \frac{1}{2\pi i} \ln \rho(\gamma_j)$ with eigenvalues λ_k^j satisfying the conditions:

$$0 \leq Re\lambda_k^j < 1. \tag{3}$$

Denote by $X_m = X \setminus S$. Let $\tilde{X} \to X_m$ be the universal covering map of X_m. Then it is a bundle with fiber $\pi_1(X_m, z_0)$, where $z_0 \in X_m$ and $\pi_1(X_m, z_0)$ denotes the fundamental group of the manifold X_m. The group $\pi_1(X_m, z_0)$ is isomorphic to the group of deck transformations of this covering and therefore acts on $\tilde{X}$ (see [5]).

It is known that there exists a basis (s.c. *Levelt basis*) of the solution space of system (1) such that the fundamental matrix of solutions $F(\widetilde{z})$ at the singular point s_j admits the following representation

$$F_j(\widetilde{z}) = H_j(z)(z - s_j)^{\Lambda^j} (\widetilde{z} - s_j)^{E_j}, \tag{4}$$

where $\widetilde{z}$ denotes the coordinates on the universal covering, $H_j(z)$ is holomorphic invertible at s_j, $\Lambda^j = diag(\kappa_1^j, ..., \kappa_n^j), \kappa_1^j \geq ... \geq \kappa_n^j$, is the diagonal matrix, with integer entries and E_j are upper triangular matrices defined above, satisfying the conditions (3).

Consider the trivial principal bundle $\tilde{X} \times GL_n(\mathbb{C}) \to \tilde{X}$ (or vector bundle $\tilde{X} \times \mathbb{C}^n \to \tilde{X}$). The quotient space $\tilde{X} \times GL_n(\mathbb{C})/\sim$ gives the locally trivial bundle on X_m, where $\sim$ is an equivalence relation identifying the pairs $(\tilde{x}, g)$ and $(\sigma\tilde{x}, \rho(\sigma)g)$, for every $\tilde{x} \in \tilde{X}, g \in GL_n(\mathbb{C})$ (or $g \in \mathbb{C}^n$). Denote the obtained bundle by $\mathbf{P}_\rho \to X_m$ (or $\mathbf{E}_\rho \to X_m$) and call it the bundle associated with the representation ρ. Obviously, this bundle according to the transformation, functions may be constructed in the following manner (see [14]).

Let $\{U_\alpha\}$ be a simple covering of X_m, i. e. every intersection $U_{\alpha_1} \cap U_{\alpha_2} \cap ... \cap U_{\alpha_k}$ is simply connected. For each U_α we choose a point $z_\alpha \in U_\alpha$ and join z_0 and z_α by γ_α starting at z_0 and ending at z_α. For a point $z \in U_\alpha \cap U_\beta$ we choose a path $\tau_\alpha \subset U_\alpha$ which starts at z_α and ends at z. Consider

$$g_{\alpha\beta}(z) = \rho\left(\gamma_\alpha \tau_\alpha(z) \tau_\beta^{-1}(z) \gamma_\beta^{-1}\right). \tag{5}$$

We see that $g_{\alpha\beta}(z) = g_{\beta\alpha}(z)$ on $U_\alpha \cap U_\beta$ and $g_{\alpha\beta} g_{\beta\gamma}(z) = g_{\alpha\gamma}(z)$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

The cocycle $\{g_{\alpha\beta}(z)\}$ does not depend on the choice of z. Hence from this cocycle we obtain a flat vector (or principal) bundle on X_m, which is denoted by $\mathbf{P}'_\rho$ ($\mathbf{E}'_\rho$.) Let $\{t_\alpha(z)\}$ be a trivialization of our bundle, i. e.

$$t_\alpha : p^{-1}(U_\alpha) \to GL_n(\mathbb{C})$$

is a holomorphic mapping. Consider the matrix-valued 1-form $\{\omega_\alpha\}$:

$$\omega_\alpha = -t_\alpha^{-1} dt_\alpha.$$

$\{g_{\alpha\beta}(z)\}$ are constant on the intersection $U_\alpha \cap U_\beta$ and $g_{\alpha\beta}(z) t_\beta(z) = t_\alpha(z)$, so on $U_\alpha \cap U_\beta$ the identity $\omega_\alpha = \omega_\beta$ holds. Indeed, replacing t_β by $t_\beta^{-1} g_{\alpha\beta}$ in the expression $\omega_\beta = -t_\beta^{-1} dt_\beta$, we obtain

$$\omega_\beta = -t_\alpha^{-1} g_{\alpha\beta}(z)\, dt_\alpha g_{\alpha\beta}^{-1}(z) = -t_\alpha^{-1} dt_\alpha.$$

So, $\omega = \{\omega_\alpha\}$ is a holomorphic 1-form on X_m and therefore is a connection 1-form of the bundle $\mathbf{P}'_\rho \to X_m$. The corresponding connection is denoted by ∇'. We will extend the pair $(\mathbf{P}'_\rho, \nabla')$ to X. As the required construction is of local character, we shall extend $\mathbf{P}'_\rho \to X_m$ to the bundle $\mathbf{P}''_\rho \to X_m \cup \{s_i\}$, where $s_i \in S$.

First consider the extension of the principal bundle $\mathbf{P}'_\rho \to X_m$. Let a neighborhood V_i of the point s_i meet $U_{\alpha_1}, U_{\alpha_2}, ... U_{\alpha_k}$. As we noted when constructing the bundle from transition functions (5) only one of them is different from identity. Let us denote it by g_{1k}, then $g_{1k} = M_i$, where M_i is the monodromy which corresponds to the singular point s_i and is obtained by representation (2). Select a branch of the multivalued function $(\tilde{z} - s_i)^{E_i}$ containing the point $\tilde{s}_i \in \tilde{U}_i$. Thus the selected branch defines a function

$$g_{01} = (z - s_i)^{E_i}. \tag{6}$$

Denote by g_{02} the extension of g_{01} along the path which goes around s_i counterclockwise, and similarly for other points. At last on $U_i \cap U_{\alpha_k} \cap U_{\alpha_1}$ we shall have:

$$g_{0k}(z) = g_{01}(z) M_i = g_{01}(z) g_{0k}(z).$$

The function $g_{0k} : V_i \to GL_n(\mathbb{C})$ is the one defined at the point s_i, and takes its value coinciding with the monodromy matrix. It means, that we made an extension of the bundle to the point s_i. In a neighborhood of s_i one will have

$$\omega_i = dg_{0k} g_{0k}^{-1} = E_i \frac{dz}{z - s_i}. \tag{7}$$

So we obtained the holomorphic principal bundle $\mathbf{P}_\rho \to X$ on the surface X. The vector bundle associated to $\mathbf{P}_\rho \to X$, which we denote by $\mathbf{E}_\rho \to X$ is not topologically trivial and the Chern number $c_1(\mathbf{E}_\rho)$ of $\mathbf{E}_\rho \to X$ is equal to

$$c_1(\mathbf{E}_\rho) = \sum_{i=1}^{m} tr(E_i). \tag{8}$$

Matrix valued meromorphic 1-form $\omega = (\omega_i)$ is a connection form of the logarithmic connection ∇ of $\mathbf{E}_\rho$. From this the holomorphic sections of $\mathbf{E}_\rho$ are the solutions of the equation

$$\nabla f = 0 \iff df = \omega f, \tag{9}$$

and if the Riemann surface X is the Riemann sphere $\mathbb{CP}^1$, then

$$\omega = \sum_{j=1}^{m} \frac{A_j}{z - s_j} dz. \tag{10}$$

Let $\beta_i^j = \kappa_i^j + \lambda_i^j$. The numbers β_i^j, κ_i^j will be called *exponents* (or j-exponents) and *valuations* of the solution space of the system (1) at the point s_i, respectively. Besides, $\sum_{j=1}^{m}\sum_{i=1}^{n}\beta_i^j = 0$ and $\kappa_i^j = [Re\beta_i^j]$, where $[Re\beta_i^j]$ denotes an integer part of real numbers $[Re\beta_i^j]$ (see [4]).

An important characteristic of the behavior of a solution of system (9) in a neighborhood of a singular point s_j is the integer part of the real part of the number β_i^j, which equals to κ_i^j; clearly these also influence behavior of the sections of the bundle induced by the system of equations.

Above we have constructed the canonical extension of the bundle. In this case we take $\Lambda^j = (0, ..., 0)$ and M_j are upper triangular matrices. In the general case, for every given date (S, ρ) induced from system (1), there exists a family of the holomorphic vector bundles with logarithmic connection $(\nabla^{C,\Lambda}, E_\rho^{C,\Lambda})$. All the elements from this family depend on the collection of the matrices $C = (C_1, ..., C_m)$ such that $C_j^{-1} M_j C_j$ are upper triangular and the collection of diagonal matrices of the *valuations* $\Lambda = (\Lambda^1, ..., \Lambda^m)$, where $\Lambda^j = diag(\kappa_1^j, ..., \kappa_n^j)$, as mentioned above, are diagonal matrices with integer entries, satisfying the condition $\kappa_1^j \geq ... \geq \kappa_n^j$. Reduction of the monodromy matrices M_j to the upper triangular form by C_j is the action C_j on the solution space such that

$$g_{01} \to \tilde{g}_{01} = (z - s_j)^{\Lambda^j} (z - s_j)^{C_i^{-1} E_j C_j} C_j^{-1},$$

$$\omega_j \to \tilde{\omega}_j = \left(\Lambda^j + (z - s_j)^{\Lambda^j} C_j^{-1} E_j C_j (z - s_j)^{-\Lambda^j}\right) \frac{dz}{z - s_j},$$

where g_{01} is a cocycle defined by formula (6) and ω_j is defined by the expression (7). For the canonical extension from this family, we use the notation (E^0, ∇^0) and as mentioned above this corresponds to $\Lambda = (\Lambda^0, ..., \Lambda^0)$, where $\Lambda^0 = (0, ..., 0)$. The canonical extension does not depend on the collection of matrices C.

According to Birkhoff-Grothendieck theorem $E^{C,\Lambda}$ decomposes into a direct sum of line bundles (see [22]):

$$E(k_1^{C,\Lambda}) \oplus \cdots \oplus E(k_n^{C,\Lambda}) \to \mathbb{CP}^1,$$

where $E(k_j^{C,\Lambda})$ here and below denotes line bundle (one dimensional) with Chern number $k_j^{C,\Lambda}$.

The integer valued vector $K^{C,\Lambda} = (k_1^{C,\Lambda}, ..., k_n^{C,\Lambda})$ is called the *splitting type* of the vector bundle $E^{C,\Lambda}$.

For the canonical extension, we omit upper indices and for such holomorphic vector bundle we use the decomposition of the form

$$E \cong E(k_1) \oplus \cdots \oplus E(k_n). \tag{11}$$

It is known that if the representation is irreducible then

$$\sum_{i=1}^{n}(k_1 - k_i) \leq \frac{1}{2}n(n-1)(m-2). \tag{12}$$

In particular, the inequality (12) implies that the irreducibility properties of Riemann (hypergeometric) equations lead to the stability of the corresponding canonical bundle.

2. Loop space and deformation of holomorphic vector bundles

In this section, we consider the deformation of complex structures of Kodaira-Spenser [19] of canonical bundles induced from the Fuchsian system. For the notation and basic facts from this theory, we refer to paper [24], adapted for our application.

Let Γ be a smooth closed positively oriented loop in $\mathbb{CP}^1$ which separates $\mathbb{CP}^1$ into two connected domains U_+ and U_-. Suppose $0 \in U_+$ and $\infty \in U_-$. Let us denote by Ω the space of all Hölder-continuous matrix functions $f : \Gamma \to GL_n(\mathbb{C})$ with the natural topology. For every matrix function $f \in \Omega$ we have the following formula for the *global index* k of f:

$$k = \frac{1}{2\pi}\Delta_\Gamma \mathtt{argdet} f(t).$$

Let $\Omega^{\pm}$ be a space of matrix functions $f \in \Omega$ such that f is the boundary value of the matrix function holomorphic in $U^{\pm}$, respectively, and $f(\infty) = \mathbf{1}$.

It is known that any matrix function $f \in \Omega$ can be represented as

$$f(t) = f^-(t) d_K f^+(t), \tag{13}$$

where $f^{\pm} \in \Omega^{\pm}$ and d_K is a diagonal matrix $d_K = \mathtt{diag}(t^{k_1}, ..., t^{k_n})$ satisfying the condition $k_1 \geq ... \geq k_n$. The integers $k_1, ..., k_n$ will be called partial indices of f and the integer valued vector $K = (k_1, ..., k_n)$ will be called characteristic multi-index [20]. Two matrix functions $f, g \in \Omega$ will be called equivalent, if f and g have identical characteristic multi-indices [3], [22].

For $K = (k_1, k_2, ..., k_n)$, denote by Ω_K the set of equivalence classes of matrix functions from Ω, with characteristic multi-index K. The matrix-function $f \in \Omega_K$ is called *stable*, if all matrix functions have the same partial indices in a sufficiently small neighborhood of f. It is known that f is stable iff $k_1 - k_n \leq 1$. Topological space Ω decomposes into a countable number of open components

$$\Omega^k = \{f \in \Omega, \Delta_\Gamma \mathtt{argdet} f(t) = 2\pi k\}, \Omega = \cup_k \Omega^k, k \in \mathbb{Z}.$$

One has the stratification of Ω^k by the strata Ω_K, i.e., $\Omega^k = \cup_K \Omega_K$, where Ω^k is connected, and the matrix functions $f_1(t)$ and $f_2(t)$ belong to the same Ω^k if and only if $f_1(t)$ and $f_2(t)$ are homotopic.

Deformation $\Omega_{K'}$, $K^{'} = (k^{'}_1, k^{'}_2, ..., k^{'}_n)$ of the strata Ω_K, $K = (k_1, k_2, ..., k_n)$, is called elementary if $k^{'}_i = k_i$ except for two indices p and q, $p < q$, for which we have $k^{'}_p = k_p - 1$, $k^{'}_q = k_q + 1$. It follows from this that for every k, there exists a diagonal matrix in Ω^k with stable partial indices $(p+1, p+1, ..., p+1, p, p, ..., p)$, where $k = np + r$, $0 \leq r < n$ and every matrix function can be transformed into such stable (i.e.$|k_i - k_j| \leq 1$, $i, j = 1, 2, ..., n$) diagonal matrix by elementary operations. Besides, the multi-index K as a function of $f \in \Omega^k$ has discontinuities only on the strata Ω_K.

The Banach Lie group $\Omega^+ \times \Omega^-$ acts analytically on Ω via

$$f \overset{\alpha}{\longmapsto} h_1 f h_2^{-1}, f \in \Omega, h_1 \in \Omega^+, h_2 \in \Omega^-.$$

It can be readily seen that the orbit of the diagonal matrix d_K by the action α is Ω_K [22].

The stability subgroup H_K of f under the action α consists of those pairs (h_1, h_2) of upper triangular matrix-functions where the (i, j)-th entry in h_1 is a polynomial in z of degree at most $(k_1 - k_2)$ and $f = h_1 f h_2^{-1}$; the space H_K has finite dimension

$$\mathtt{dim} H_K = \sum_{k_i \geq k_j} (k_i - k_j + 1). \tag{14}$$

The stratum Ω_K is a locally closed analytical submanifold of Ω and codimension of Ω_K in Ω is equal to

$$\mathtt{dim}\Omega/\Omega_K = \sum_{k_i > k_j} (k_i - k_j - 1). \tag{15}$$

A connection between partial indices of the matrix functions and the splitting type of the holomorphic vector bundle E are presented in the following theorem:

Theorem 1. *(see [3], [22]) There is a one-to-one correspondence between the strata Ω_K and holomorphic vector bundles on $\mathbb{CP}^1$.*

It follows from this that the diagonal matrix function $d_K \in \Omega_K$ is a cocycle, which defines a holomorphic vector bundle $E(k_1) \oplus ... \oplus E(k_n)$ on $\mathbb{CP}^1$.

The holomorphic type of the bundle is determined by a vector of integers $(k_1, \dots, k_n) \in \mathbb{Z}^n$, $k_1 \geq \dots \geq k_n$, or else by a loop

$$d_K : S^1 \to GL_n(\mathbb{C}),$$

$d_K(z) = \mathtt{diag}(z^{k_1}, ..., z^{k_n})$ and S^1 is a unit circle in a complex plane. Following Theorem 1, we identify invertible matrix functions with characteristic multi-index K and holomorphic bundles with splitting type K. A holomorphic deformation of the bundle E in the parameter space $\mathcal{B}^l \subset \mathbb{C}^l$ is called a matrix function

$$D : (\mathbb{CP}^1 \setminus \{0, \infty\}) \times \mathcal{B}^l \to GL_n(\mathbb{C}),$$

holomorphic in the variable z and parameter t. Therefore, if the matrix function $D_t(z)$, $z \in \mathbb{CP}^1 \setminus \{0, \infty\}$, $t \in \mathcal{B}^l$, which depends on the parameter t, is a deformation of the vector bundle E with splitting type $(k_1, ..., k_n)$, then $D_0(z) = d_K(z)$. Here $\mathcal{B}^l \subset \mathbb{C}^l$ is an open ball containing the origin.

Let us denote by $T\mathcal{B}^l$ the tangent space of $\mathcal{B}^l$ at 0. Consider $H^1(\mathbb{CP}^1; \mathcal{O}(\mathtt{End}\ E))$ — the first cohomology group with coefficients in holomorphic sections of the bundle $\mathtt{End}(E)$. Since $\mathtt{End}(E) \cong E \otimes E^*$, the corresponding loop will be

$$d_K \otimes d_K^{-1} : S^1 \to \mathtt{GL}_{n^2}(\mathbb{C}),$$

$$d_K \otimes d_K^{-1} = \mathtt{diag}(z^{k_1-k_1}, z^{k_1-k_2}, ..., z^{k_n-k_n}).$$

Let us use the Birkhoff-Grothendieck theorem:

$$\mathtt{End}(E) = E(k_1 - k_1) \oplus E(k_1 - k_2) \oplus \cdots \oplus E(k_{n-1} - k_n) \oplus E(k_n - k_n).$$

Since

$$\dim H^1(\mathbb{CP}^1; \mathcal{O}(\mathtt{End}(E))) = \sum \mathtt{dim} H^1(\mathbb{CP}^1; \mathcal{O}(E(k_i - k_j))),$$

and

$$\dim H^1(\mathbb{CP}^1; \mathcal{O}(E(k))) = |k| - 1$$

for $k < 0$, whereas

$$H^0(\mathbb{CP}^1; O(\mathtt{End}(E(k)))) = 0$$

for $k \geq 0$, using moreover the Riemann-Roch theorem one obtains

$$\mathtt{dim} H^1(\mathbb{CP}^1; \mathcal{O}(\mathtt{End} E)) = \sum_{k_i > k_j} (k_i - k_j - 1). \qquad (16)$$

Suppose $k_1 > \cdots > k_n$. Then, following [19],

$$\tilde{m}(E) = \mathtt{dim} H^1(\mathbb{CP}^1; \mathcal{O}(\mathtt{End}(E))) = \sum_{j>i} (k_i - k_j) - \frac{n(n-1)}{2},$$

and

$$m(E) = \tilde{m}(E) + \frac{n(n-1)}{2}. \qquad (17)$$

The number $m(E)$ is called *the reduced dimension of the deformation space.*

It is known that for a holomorphic bundle $E \to \mathbb{CP}^1$ there exists a complete and effective deformation, which means that the Kodaira-Spencer map

$$\tau : T\mathcal{B}^l \to H^1(\mathbb{CP}^1; \mathcal{O}(\mathtt{End}(E)))$$

defined by the formula $\tau(t) = \frac{\partial}{\partial t}\Phi(z,t) \cdot \Phi(z,t)^{-1}$ is an isomorphism. This implies that the dimension of the deformation space is equal to the number $m(E)$.

The preceding discussion established the following theorem.

Theorem 2. *Let E be a canonical vector bundle induced from Fuchsian system (23). Then dimension of the deformation space of complex structures $m(E)$ is expressed by the formula (16) and can be calculated algorithmically.*

We provide the algorithm for calculating $m(E)$ in the next section.

Example 1. For three- and two-dimensional vector bundles, the aforementioned global invariants of the bundle are sufficient for expressing the splitting type explicitly. For example, suppose $E \to \mathbb{CP}^1$ is a three-dimensional bundle with $E = E(k_1) \otimes E(k_2) \otimes E(k_3)$, $k_1 > k_2 > k_3$. Then the splitting type of this bundle is expressed in terms of the dimension of the deformation space, Chern number and Fuchs weight [5] of the bundle by the following formula:

$$k_1 = \frac{1}{3}c_1(E) + \frac{1}{3}w(E);$$
$$k_2 = \frac{1}{3}c_1(E) + \frac{1}{2}m(E) - \frac{2}{3}w(E);$$
$$k_3 = \frac{1}{3}c_1(E) - \frac{1}{2}m(E) + \frac{1}{3}w(E).$$

Indeed, for the Chern number one has the equality

$$c_1(E) = k_1 + k_2 + k_3. \tag{18}$$

For the Fuchs weight one has the equality

$$w(E) = 3k_1 - c_1(E). \tag{19}$$

The reduced dimension of the deformation space equals

$$m(E) = \mathtt{dim}H^1(\mathbb{CP}^1; \mathcal{O}(\mathtt{End}(E))) + 3\sum_{k_i > k_j}(k_i - k_j - 1) + 3 \tag{20}$$

$$= k_1 - k_2 - 1 + k_1 - k_3 - 1 + k_2 - k_3 - 1 + 3 = 2k_1 - 2k_3.$$

Solving the system of equations (18), (19), (20) we obtain the above formulas for a splitting type of canonical vector bundle. Formulas of such type do not exist in higher dimensions. In the next section, we use another approach to calculate the splitting type of canonical vector bundles, and as a consequence, we obtain formulas for computing the dimension of the space of complex structures.

3. Computation of the dimension of the moduli space of holomorphic structures of the canonical bundle

Here, we summarize some basic facts from above as the following, one possible version of the Riemann-Hilbert monodromy problem, which is known as a *Riemann-Hilbert problem with given asymptotic* [4]:

For given representation

$$\rho : \pi_1(X_m, z_0) \to GL_n(\mathbb{C}) \tag{21}$$

and collection of diagonal matrices $\Lambda = (\Lambda^1, ..., \Lambda^m)$ such that

$$\sum_{k=1}^{m} \Lambda^k = 0 \tag{22}$$

construct the Fuchsian system

$$dF = \left(\sum_{j}^{m} \frac{A_j}{z - s_j} dz\right) F, \tag{23}$$

where $A_j, j = 1, ..., m$ are constant matrices: $A_j = res_{z=s_j}\omega(z)$ and $\sum_{j=1}^{m} A_j = 0$ if $s_j \neq \infty$, for all $j = 1, ..., m$, with given monodromy ρ and given Λ as collection of matrices of valuations.

In general, for the class of the Fuchsian systems, this problem has negative solution but it is known that there exists a system of differential equations

$$\frac{dy(z)}{dz} = B(z)y(z) \tag{24}$$

with $\Lambda = (\Lambda^1, ..., \Lambda^m)$ given matrix of valuations, whose monodromy representation coincides with ρ, is Fuchsian at the given points $s_1, ..., s_m$ and have apparent singularity at ∞.

The existence system (24) with given monodromy and singular points $s_1, ..., s_m$, follows from positive solution of Riemann-Hilbert monodromy problem for regular systems (Plemelj theorem [5]).

At the same time for any system of equations (23) and collection of diagonal matrices $\Lambda^1, ..., \Lambda^m$ with integer entries there exists the rational matrix function $T(z)$ such that the system of equations (23) is gauge equivalent to the system

$$\frac{d\tilde{y}}{dz} = B(z)\tilde{y}, \quad B(z) = T(z)\left(\sum_{j=0}^{m} \frac{A_j}{z - s_j}\right)T^{-1}(z) + \frac{dT(z)}{dz}T^{-1}(z) \tag{25}$$

with valuations $\Lambda^1, ..., \Lambda^m$ at the singular points $s_1, ..., s_m$, respectively, and possible apparent singularity at ∞ [14].

We call rational matrix function $T(z)$ a *transform matrix* [14].

Proposition 1. [14] *The transform matrix $T(z)$ is a meromorphic matrix function, and it can be constructed algorithmically.*

Consider canonical extension (E^0, ∇^0) of the holomorphic vector bundle induced from the system (24) and suppose that splitting type of this bundle is $K = (k_1, ..., k_n), k_1 \geq ... \geq k_n$:

$$E^0 \cong E(k_1) \oplus ... \oplus E(k_n).$$

As mentioned above the canonical extension is induced from the regular system of the form (24) with valuations $\Lambda^j = (0, ..., 0)$ at each singular points s_j, $j = 1, ..., m$.

Proposition 2. [4], [14] *The splitting type of the canonical vector bundle E^0 coincides with the partial indices of the transform matrix $T(z)$.*

From Proposition 1 and Proposition 2, it follows that the splitting type of the canonical vector bundle can be calculated algorithmically. Therefore, the invariants (14),(15) and (20) can be calculated algorithmically as well. This is the complete proof of Theorem 2.

We provide a brief description of the algorithm for constructing the transformation matrix here.

Step 1: Choose any coefficient matrix of the system and reduce it to its Jordan canonical form.

Step 2: Arrange the real parts of the eigenvalues in descending order.

Step 3: Equate eigenvalues to the corresponding vector of valuations.

Remark: For the canonical extension, the vector of valuations has coordinates set to 0.

Step 4: Repeat Steps 1-3 for all coefficient matrices.

Remark. The coefficient matrix changes after each step.

Each step is carried out using a matrix. *The transform matrix is the product of these matrices.*

For calculating the partial indices of the transform matrix, we use the algorithm and its program implementation provided in [1] and [2].

4. Conclusion

Based on our previous papers [14], [15] we present a computer implementation of the algorithm for constructing the transform matrix. As a consequence of this, we obtain a method for calculating the dimension of the deformation space of complex structures of the canonical vector bundles induced from the Fuchsian system by the splitting type of this bundle. As demonstrated by formulas (14) and (15), and Example 1, this number is a fundamental invariant of the holomorphic vector bundles.

Acknowledgments. Research supported by Shota Rustaveli National Scientific Foundation of Georgia, Grant Agreement FR22-354.

Availability of data and materials. Program realization of the algorithm available at the address: https://github.com/giorgi94/fuchsian–toolkit.

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Received 30.09.2024; accepted 20.11.2024.

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