

EXTREMAL PROBLEMS FOR STEINER 3-CHAINS

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Abstract. Solutions of several extremal problems for Steiner chains of three circles are presented. More concretely, the poristic Steiner chains of three circles with an extremal sum of radii are identified and the corresponding extremal values are expressed in terms of the radii of Soddy circles of the Steiner chain considered. Analogous results are given for the sum of areas of Steiner chains of three circles. We also compute the extremal values of the third momentum of curvatures of poristic Steiner chains. The proofs of all these results are based on an interesting interpretation of the Descartes circle theorem given in a recent paper of R. Schwarz and S. Tabachnikov.

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1. Introduction

We discuss several extremal problems for the so-called Steiner chains consisting of three circles [2]. Recall that the term *Steiner chain* refers to a collection of tangent circles in Euclidean plane each of which is tangent to fixed two nested circles called the *Soddy circles* of the Steiner chain considered [2]. Steiner chains were discussed in many papers, in particular, in the context of *Steiner porism* (see, e.g., [10]).

In this paper we only consider Steiner chains consisting of three circles and call them Steiner 3-chains. Thus in the sequel the term *Steiner 3-chain* refers to a triple of pairwise externally tangent circles in Euclidean plane (such triples of circles are sometimes called “kissing circles”). A classical result of Apollonius states that, for any triple of “kissing circles”, there exist exactly two circles which are tangent to each of the three given circles. These two circles are called the *Soddy circles* of a given Steiner 3-chain. The well known Descartes circle theorem enables one to compute the curvatures of Soddy circles in terms of the curvatures of the three given “kissing circles” [2].

The extremal problems considered in the sequel arise in the context of the so-called *Steiner porism* [2]. Its special case we are interested in, states that there exists a one-dimensional family of Steiner 3-chains of circles having the same Soddy circles [10]. By a way of analogy with the Poncelet porism [2] the collection of Steiner 3-chains with fixed Soddy circles will be called *poristic Steiner 3-chains*. This point of view suggests several extremal problems for poristic Steiner 3-chains with the fixed Soddy circles. As usual,

the most natural extremal problems are concerned with the sum of radii and the sum of areas of poristic Steiner 3-chains. The main aim of the present paper is to present solutions to these and a few other extremal problems.

The first main result, Theorem 1, presents explicit formulas for the extremal values of the sum of radii and characterization of extremal chains. The second main result, Theorem 2, gives analogous formulas for the sum of areas of poristic Steiner 3-chains. We also give a similar result for the third momentum of curvatures of poristic Steiner 3-chains (Theorem 3). The proofs of all these results are based on the invariance of the first two momenta of curvatures of poristic Steiner 3-chains (see, e.g., [10]).

It should be noted that a special case of the extremal problem for the sum of areas was discussed in a prize-winning paper of K.Kirajiev [7]. The answer given in [7] is correct but the corresponding reasoning is definitely incomplete. As an application of our approach we explain how the proof of Theorem 2 given above can be modified to provide a rigorous proof of the mentioned result from [7]. Certain extremal problems for constrained chains of tangent circles have also been discussed in [3], [4] but the setting and context in those papers are essentially different from ours. In the last section, we briefly discuss some further connections and feasible generalizations of our results and approach.

2. Descartes circle theorem and Steiner porism

Consider a system Γ consisting of three circles Γ_i of radii r_i in Euclidean plane which are pairwise externally tangent, and denote by $b_i = \frac{1}{r_i}$ their curvatures. Such a triple of circles are said to form a *Steiner 3-chain*. An ancient result of Apollonius states that there are exactly two other circles, nowadays called the *Soddy circles* of Γ , each of which is tangent to the three given circles. One of Soddy circles is externally tangent to the given ones and lies in the inner (bounded) component of the complement to the union of closures of their interiors. Let us denote its radius by r and its curvature by $a_1 = 1/r$. The second Soddy circle contains the given ones in its interior. Let us denote its radius by R and by $a_2 = -1/R$ its *signed curvature*.

The so-called Descartes circle theorem states that both a_1 and a_2 satisfy the following relation

$$(a + b_1 + b_2 + b_3)^2 = 2(a^2 + b_1^2 + b_2^2 + b_3^2). \quad (1)$$

Given a Steiner 3-chain with curvatures b_i , we may interpret equation (1) as a quadratic equation for a of the form:

$$a^2 - 2(b_1 + b_2 + b_3)a + [(b_1^2 + b_2^2 + b_3^2) - 2(b_1b_2 + b_2b_3 + b_3b_1)] = 0. \quad (2)$$

Solving (2) yields explicit formulas for the (signed) curvatures a_1 and a_2 of Soddy circles in terms of b_i :

$$a_1 = b_1 + b_2 + b_3 + 2\sqrt{D}, a_2 = b_1 + b_2 + b_3 - 2\sqrt{D}, \quad (3)$$

where $D = b_1b_2 + b_2b_3 + b_3b_1$, which yields explicit formulas for the radii of Soddy circles:

$$r = \frac{1}{a_1}, R = -\frac{1}{a_2}.$$

From (1) it also follows

$$D = \frac{(a_1 - a_2)^2}{16}.$$

So in terms of Soddy radii we get

$$D = \frac{(R + r)^2}{16R^2r^2}.$$

It is also known (see, e.g., [8]) that the distance d between the centres of Soddy circles is related to their radii r, R by

$$d = \sqrt{R^2 - 14Rr + r^2}. \quad (4)$$

The proofs of all these results can be obtained using an inversion which takes a given pair of nested circles to a pair of concentric ones (see, e.g., [8]). Actually, formula (4) yields a criterion of Soddy circles. Namely, two nested circles with radii $0 < r < R$ and distance d between their centres, are the Soddy circles of a certain Steiner 3-chain if and only if equation (4) is satisfied.

Example 1. (classical) For three equal kissing circles of radius $k = 1$, we have $b_i = 1, \sum b_i = 3, D = 3$ so $a_1 = 3 + 2\sqrt{3}, a_2 = 3 - 2\sqrt{3}$. Notice that $a_2 < 0$. Then $r = 1/(2\sqrt{3} + 3), R = 1/(2\sqrt{3} - 3)$. So formula (4) gives

$$d = \sqrt{\frac{1}{(3 - 2\sqrt{3})^2} - \frac{14}{(3 - 2\sqrt{3})^2(3 + 2\sqrt{3})^2} + \frac{1}{(3 + 2\sqrt{3})^2}} = 0,$$

as was expected since in this case the Soddy circles are obviously concentric.

Example 2. (symmetric) For a Steiner 3-chain with radii $r_1 = 20, r_2 = r_3 = 10$ we have $b_1 = 0.05, b_2 = b_3 = 0.1$. So $\sum b_i = 0.25$ and $D = 0.2$ and from formulas (3) we get that $a_1 = 0.532, a_2 = -0.032$. It follows that the radii of Soddy circles are $r = 1.880, R = 31.25$ and the distance between their centres is $d = 12.55$ which can be easily verified geometrically.

This example can be considered as a typical one since it is easy to realize that any poristic Steiner family contains a pair of such symmetric Steiner 3-chains. To see this just take the center of one poristic circle on the line passing through the centres of Soddy circles called the Soddy line.

Remark 1. In both examples above we considered a given Steiner 3-chain and used Descartes circle theorem to determine the Soddy circles. In

the next example we start with a pair of circles satisfying (4) and compute the curvatures of symmetric poristic Steiner 3-chains.

Example 3. For $R = 30, r = 2$, formula (4) gives $d = 8$. Placing the centres of virtual Soddy circles at the origin and point $(8, 0)$ one easily finds the curvatures of symmetric Steiner 3-chains. Indeed, a circle of radius 10 centered at $(20, 0)$ is tangent to the both Soddy circles. It is then easy to show that the two circles of radii 20 centered at the bisectors of the second and third quadrant belong to the sought Steiner 3-chain. In this case, we have $a_1 = 0.5, a_2 = -1/30 = -0.033(3), b_1 = 0.1, b_2 = b_3 = 0.05$ so that relation (1) holds true.

Remark 2. The same considerations are valid for any triple of positive numbers (R, r, d) satisfying (4), which shows that (4) is indeed a criterion of Soddy circles.

Remark 3. The set of triples of positive numbers satisfying (4) can be identified with the intersection of ellipsoid given by equation (4) with $\mathbb{R}_+^3$. An analogous interpretation is possible for the set of positive pairs (A, B) which are equal to the first two momenta curvatures of poristic Steiner 3-chains.

For brevity, any pair of nested circles satisfying (4) will be called *Soddy circles* and the triple (R, r, d) will be called its *gauge*. Clearly, a pair of nested circles is determined by its gauge up to an isometry of the ambient plane. For a Steiner 3-chain, we define its *gauge* as the gauge of its Soddy circles.

In the sequel we work in the framework of *Steiner porism* [8] and freely use related concepts and results. In our setting, Steiner porism states that there exists a whole one-dimensional family of Steiner 3-chains with the same Soddy circles [2]. The members of the aforesaid family will be called *poristic Steiner 3-chains*. Clearly, all invariants of the family of poristic Steiner 3-chains are uniquely determined by its gauge.

This context suggests several extremal problems three of which are discussed below. For example, one can search for the maximum and minimum of the sum of radii of Steiner 3-chains with fixed Soddy circles. One of our main aims is to characterize the extremal Steiner 3-chains and express the corresponding extremal values in terms of the gauge of poristic family considered.

Remark 4. A similar problem for the sum of areas was considered in [7] in one concrete case. The answer given in [7] is correct but the proof is only outlined and no comments are given on how it can be made completely rigorous.

For brevity and convenience we introduce some definitions and terms. The *extent* $L(\Gamma)$ of a Steiner 3-chain Γ is defined as the sum of radii of its circles. Analogously, the *content* $S(\Gamma)$ of a Steiner 3-chain is defined as the sum of radii of its circles. These are analogs of the perimeter and sum of areas respectively. For a natural k , the k -th momentum of curvatures of a Steiner 3-chain Γ will be denoted by $M_k(\Gamma)$. Notice that the centres

of a triple of kissing circles constructed in Example 2 form an isosceles triangle. For a given Soddy gauge (R, r, d) , there exist exactly two Steiner 3-chains with the same property. Such Steiner chains will be referred to as *symmetric Steiner chains*. A lot of information on poristic Steiner 3-chains can be obtained by studying the symmetric Steiner chains.

To show some connections and possible applications of our results we add two remarks on similar problems in the framework of *Poncelet porism* [2].

Remark 5. It is known that the centres of all circles in poristic Steiner 3-chains lie on an ellipse, called *Soddy ellipse*, and the three points of tangency lie on a circle (called the *socle* of poristic family). It is easy to calculate the sizes and location of the Soddy ellipse and socle. What is really remarkable, the sides of the triangle formed by centres of Steiner 3-chain is tangent to the socle. In other words, the Soddy ellipse and socle are a pair of quadrics satisfying Poncelet porism [2]. Extremal problems for the Poncelet porism were studied, in particular, in [6], [9], which suggested a paradigm which we use below.

Remark 6. A special case of Poncelet porism is concerned with the so-called *bicentric polygons* which interscribe a given pair of nested circles [2]. We note that certain extremal problems for poristic bicentric polygons have been studied in a series of papers of M.Radic (see, e.g. [9]). Taking into account that the both mentioned porisms are in a certain sense equivalent (see, e.g., [1]) it was natural to suppose that similar results can be obtained for Steiner porism, too. This conjecture together with some helpful hints was formulated by G.Khimshiashvili at a doctoral seminar at Ilia State university, which enabled the authors to obtain the results presented below. The both authors equally contributed to proving of the results and writing up the text of the present paper.

3. Invariants of poristic Steiner 3-chains

A crucial role in our approach is played by the two well known invariants of poristic Steiner 3-chains (see, e.g., [10]). Namely, it is easy to verify that equation (1) implies invariance of the first two momenta of curvatures b_i . We present two formulas which follow from (1) and Vietá formulas. The first one is evident so we give it without proof.

Corollary 1.

$$b_1 + b_2 + b_3 = \frac{a_1 + a_2}{2}. \quad (5)$$

The second formula is less obvious so we give its proof.

Corollary 2.

$$b_1^2 + b_2^2 + b_3^2 = \frac{a_1^2 + a_2^2 + 6a_1a_2}{8}. \quad (6)$$

Proof. Subtracting expressions (3) for a_1 and a_2 we see that $a_2 - a_1 =$

$4\sqrt{D}$. Hence

$$b_1b_2 + b_2b_3 + b_3b_1 = \frac{(a_2 - a_1)^2}{16}.$$

At the same time,

$$b_1^2 + b_2^2 + b_3^2 = (b_1 + b_2 + b_3)^2 - 2D = (a_1 + a_2)^2/4 - (a_2 - a_1)^2/8 = \frac{a_1^2 + a_2^2 + 6a_1a_2}{8},$$

which completes the proof.

For a given Soddy gauge (R, r, d) , let us put

$$A = \frac{a_1 + a_2}{2}, B = \frac{a_1^2 + a_2^2 + 6a_1a_2}{8}, \quad (7)$$

where $a_1 = 1/r, a_2 = -1/R$.

We also give an expression for the product b_2b_3 which will be repeatedly used in the sequel.

Corollary 3. With the above notations, one has

$$b_2b_3 = \frac{2b_1^2 - 2Ab_1 + A^2 - B}{2} \quad (8)$$

This follows by noticing that

$$b_2b_3 = \frac{(b_2 + b_3)^2 - (b_2^2 + b_3^2)}{2} = \frac{(A - b_1)^2 - (B - b_1^2)}{2}.$$

4. Extremal problems for poristic Steiner 3-chains

Given a pair F of Soddy circles with the gauge (R, r, d) , let us denote by b_i the curvatures of a certain poristic Steiner 3-chain of circles Γ_i associated with F . The most natural and geometrically significant target functions to be optimized are the sum of poristic radii $L = \sum r_i$ and the sum of poristic radii squared $S = \sum r_i^2$. Notice that L and S are constant multiples of the perimeter of the triangle formed by the centres of Γ_i and the sum of areas of poristic circles, respectively. Similar problems can be studied for the higher momenta of curvatures $M_k = \sum b_i^k$ with $k \geq 3$. In this paper, we find the extremal values of third and fourth momenta M_3, M_4 , which may serve as a paradigm for studying extrema of the higher momenta M_k with $k \geq 5$.

In terms of curvatures b_i the first two target functions read

$$L = \sum \frac{1}{b_i}. \quad (9)$$

$$S = \sum \frac{1}{b_i^2}. \quad (10)$$

Notice that L, S, M_k are symmetric functions of curvatures, which is essential for our approach. For further use we determine the ranges of curvatures b_i and their inverses r_i .

Proposition 1. *For a given pair of Soddy circles with gauge (R, r, d) , the minimal and maximal possible values of poristic radii r_i are*

$$r_* = \frac{R - d - r}{2}, r^* = \frac{R + d - r}{2},$$

while the minimal and maximal values of poristic curvatures are

$$b_* = \frac{2}{R + d - r}, b^* = \frac{2}{R - d - r}$$

respectively.

Let l be the line through the centres of Soddy circles called the *axis of porism*. It is geometrically obvious and easy to prove rigorously that the extremal values of poristic radii are attained at the two poristic circles having their centres on line l . The radii of those circles are readily expressed in terms of the gauge, which yields the formulas given in the lemma.

Example 3 (continued). For $r = 2, R = 30$, formula (4) gives

$$d = \sqrt{900 - 840 + 4} = \sqrt{64} = 8.$$

We introduce a Cartesian system of coordinates so that the center of outer Soddy circle is at the origin and the center of inner Soddy circle is at the point $(8, 0)$. Then it is obvious that the biggest poristic circle has radius 18 and the radius of the smallest one is 10, which of course fits the formulas given in Proposition 1. Thus in this case $b_* = 1/18 = 0.05(5) > 0.56$ and $b^* = 1/10 = 0.1$.

To study the extremal problems considered below we use the following approach. The equations (5), (6) and Newton identities for power sums imply that the values of any symmetric function f of curvatures b_i are uniquely determined by the value of $b_1 \in [b_*, b^*]$. Using formulas (5), (6), (8) and putting $x = b_1$ we can express a symmetric target function f as a function of variable x . Next, we compute derivative f' and consider its roots which yield the critical points. As we will see, for L, S, M_3 , the critical points are given by explicit algebraic formulas. So we can investigate their location and type using high school calculus. We have also check the endpoints of the curvature range b_*, b^* . It turns out that, for all three problems considered below, the critical points lie in $[b_*, b^*]$, which enables us to compute the extremal values in terms of the gauge of Soddy triple.

Remark 7. In general, there is no obvious reason why the critical points should lie inside $[b_*, b^*]$. In the example studied in [7] the extrema are attained at the end-points. So it is indeed necessary to investigate the location of critical points with respect to the segment $[b_*, b^*]$, which is done in our Lemma 2.

To illustrate the method described above we apply it first in the case where the target function P is just the product of poristic curvatures, $P = b_1 b_2 b_3$.

First, we use formula (3) to express P as a function of one variable b_1 :

$$P = b_1(b_2b_3) = b_1 \frac{2b_1^2 - 2Ab_1 + A^2 - B}{2} = \frac{2b_1^3 - 2Ab_1^2 + (A^2 - B)b_1}{2}.$$

Putting $x = b_1$ and computing derivative $P' = dP/dx$ we see that the equation for critical points is just quadratic

$$6x^2 - 4Ax + A^2 - B = 0. \quad (11)$$

It follows that the critical points of P are

$$x_{\pm} = \frac{2A \pm \sqrt{6B - 2A^2}}{6}.$$

Since

$$A = \frac{(R - r)^2}{4R^2r^2} > 0$$

we have

$$A^2 - B = \frac{(R + r)^2}{8R^2r^2} > 0.$$

So both roots are positive and our next aim is to express them through the radii R, r and determine their location with respect to the segment $[b_*, b^*]$.

To this end we observe that

$$6B - 2A^2 = \frac{R^2 - 14Rr + r^2}{4R^2r^2} > 0.$$

Since (R, r, d) is a Soddy gauge we have $R^2 - 14Rr + r^2 = d^2$ and end up with a remarkably simple expression for the critical points

$$x_{\pm} = \frac{2R - 2r \pm d}{12Rr}. \quad (12)$$

The second derivative is $P'' = 6x - 42A$ and it is elementary to verify that $P''(x_-) < 0, P''(x_+) > 0$. Hence x_- is a point of local maximum for P , which in this case is the global one. Analogously, the minimum of P is attained at x_+ .

Thus we have found the global extrema of P but it is now necessary to find the extrema in the segment $[b_*, b^*]$. To this end we prove the following proposition which is crucial for our considerations.

Proposition 2. *For any Soddy gauge (R, r, d) , the critical points $x_{\pm}$ lie into $[b_*, b^*]$.*

Proof. Let us first show

$$x_+ = \frac{2R - 2r + d}{12Rr} < b^* < \frac{2}{R - d - r}.$$

Taking the difference $x_+ - b_+$ and clearing the positive denominators we see that the above inequality is equivalent to the inequality

$$24Rr < 2R^2 + 2r^2 - 28Rr - d^2 - d(R - r),$$

which in turn is equivalent to

$$d^2 - d(R - r) > 0$$

and finally we have an inequality

$$d < R - r,$$

which is true since (R, r, d) is a Soddy triple. Thus the first inequality is proved.

An analogous argument shows that

$$x_- = \frac{2R - 2r - d}{12Rr} > b_* = \frac{2}{R + d - r},$$

which completes the proof of proposition.

This means that we have located the extrema of function P .

Proposition 3. *The extremal values of function P are attained at the points $x_{\pm}$.*

It is now easy to give formulas for the extremal values of P which we omit for the reason of space.

5. Solutions of three extremal problems for Steiner 3-chains

We use the same method to solve the three extremal problems mentioned in the Introduction. In the case of function L formulas (5), (6), (8) imply

$$L = \frac{1}{x} + \frac{2(A - x)}{2x^2 - 2Ax + A^2 - B}.$$

After computing and simplifying the derivative $L' = dL/dx$ we get

$$L' = -\frac{6(A^2 - B)x^2 - 4A(A^2 - B)x + (A^2 - B)^2}{x^2(2x^2 - 2Ax + A^2 - B)^2}. \quad (13)$$

Remarkably, the equation of critical points reduces to the same quadratic equation (11):

$$6x^2 - 4Ax + (A^2 - B) = 0.$$

So its roots are as above

$$x_{\pm} = \frac{2A \pm \sqrt{6B - 2A^2}}{6}.$$

Thus we can refer to Proposition 2, which yields our first main result.

Theorem 1. *The extremal values of L are attained at the points $x_{\pm}$ given above.*

The case of function S is investigated similarly. Using formulas (2), (3), (4), we can rewrite the target function as

$$S = \frac{1}{b_1^2} + \frac{b_2^2 + b_3^2}{b_2^2 b_3^2} = \frac{1}{b_1^2} + \frac{B - b_1^2}{(2b_1^2 - 2Ab_1 + A^2 - B)^2} = \frac{(2b_1^2 - 2Ab_1 + A^2 - B)^2 + b_1^2(B - b_1^2)}{b_1^2(2b_1^2 - 2Ab_1 + A^2 - B)^2}.$$

Omitting further routine computations we just state that it turns out that the critical points of S are given by the same quadratic equation (11), which gives our second main result.

Theorem 2. *The extremal values of S are attained at the points $x_{\pm}$ given above.*

We omit again the explicit formulas for the extremal values of S since they are obtained by a direct computation.

Finally, we solve the extremal problem for the third moment of poristic curvatures $g = M_3$. First, we express g as a function of b_1 using (5), (6), which gives

$$\begin{aligned} g &= \sum b_i^3 = b_1^3 + (b_2 + b_3)^3 - 3b_2b_3(b_2 + b_3) = \\ &= b_1^3 + (A - b_1)^3 - (A - b_1) \frac{6b_1^2 - 6Ab_1 + 3(A^2 - B)}{2} = \\ &= \frac{6b_1^3 - 6Ab_1^2 + 3(A^2 - B)b_1 - A(3B - 3A^2)}{2}. \end{aligned}$$

Since $g = M_3$ is just a cubic polynomial of $x = b_1$, its critical points are easy to compute. Namely, after some simplifications the derivative $g' = dg/dx$ becomes

$$g' = \frac{18x^2 - 12Ax + (3A^2 - 3B)}{2}.$$

Hence the critical points are again the roots $x_{\pm}$ of the same quadratic equation (11):

$$6x^2 - 4Ax + (A^2 - B) = 0.$$

Thus we can again refer to Proposition 2, which yields our third main result.

Theorem 3. *The extremal values of M_3 are attained the points $x_{\pm}$ given above.*

Analogous problems for higher moments of curvatures require solution of more complicated critical point equations and will be considered elsewhere.

6. Concluding remarks

First of all, it is natural to search for generalizations of our results to Steiner n -chains for arbitrary n . This seems feasible since, for arbitrary n ,

there is an analog of equation (4) and invariance of the first $n - 1$ momenta of curvatures [10].

It is also possible to connect Theorem 1 with similar results for Poncelet porism as follows. It is known that the triangle of centres of poristic Steiner 3-chains belongs to a poristic family of triangles interscribed between the socle and Soddy ellipse (see, e.g., [8]). Thus the solution given above could also be obtained from an analogous result for Poncelet porism which follows from the formulas given in [9]. Moreover, our results can be related to the extremal problem for an analog of Steiner chains inscribed in arbelos considered in [3].

Finally, [9] contains Fuss relations for n -pointed star polygons in Poncelet porism with $n = 5, 6, 7$. It is interesting to check if some of those Fuss relations are compatible with 4, which could give another relation to Poncelet porism (cf. [1]).

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