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HOLOMORPHIC TYPE OF VECTOR BUNDLES ON THE RIEMANN SPHERE INDUCED FROM SECOND ORDER FUCHSIAN SYSTEMS

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Abstract. For second order systems with three singular points we give complete characterization of corresponding vector bundles by invariants of the Fuchsian system, in particular, we compute splitting type of vector bundles of an explicitly form.

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1. Introduction

Let a Fuchsian system of the form

$$dF = \left(\sum_{j}^{m} \frac{A_j}{z - s_j} dz \right) F, \tag{1}$$

be given, where $S = \{s_1, s_2, ..., s_m\}$ is the set of singular points of the matrix function ω, $A_j, j = 1, ..., m$ are constant matrices: $A_j = res_{z=s_j}\omega(z)$ and $\sum_{j=1}^{m} A_j = 0$ if $s_j \neq \infty$, for all $j = 1, ..., m$ (see [4], [5], [7])

Along with the system (1) we consider $n-$order Fuchsian equation

$$f^{(n)}(z) + a_1(z) f^{(n-1)}(z) + ... + a_{n-1}(z) f^{'} + a_n(z) f(z) = 0 \tag{2}$$

on the complex plane, with same singular points, where $a_j(z), j = 0, ..., m$ are meromorphic functions satisfying Fuchsian condition with respect to orders of singularity (see [7], [12]). By definition the *exponents* of the singular point s_j are the roots of *indicial equation* and satisfy the Fuchs relation (see [3], [12], [13])

$$\sum_{j=1}^{m} (\lambda_1^j + ... + \lambda_n^j) = \frac{1}{2} n(n-1)(m-2). \tag{3}$$

The system (1) and the equation(2) define the *monodromy representation*

$$\rho : \pi_1(X_m, z_0) \to GL_n(\mathbb{C}) \tag{4}$$

by the correspondence $g_j \to M_j$, where g_j are the generators of the fundamental group and M_j are the monodromy matrices (see [3], [9], [10]).

The Fuchsian differential equation (2) with m singular pints defined by

$$p_{eq} = \frac{n^2}{2}(m-2) + \frac{nm}{2} \tag{5}$$

parameters, the monodromy representation depends on

$$p_{rep} = n^2(m-2) + 1 \tag{6}$$

parameters (see [4], [13]). The different (defect number)

$$d = \frac{nm(n-1)}{2} + (1 - n^2)$$

is always non negative and when $n = 2, m = 3$ defect number equals zero. Such equation is a Riemann equation

$$f^{''}(z) + \left(\frac{1-\rho_1-\rho_2}{z-s_1} + \frac{1-\sigma_1-\sigma_2}{z-s_2} + \frac{1-\tau_1-\tau_2}{z-s_3}\right) f^{'}(z) \tag{7}$$

$$+\left(\frac{\rho_1\rho_2(s_1-s_2)(s_1-s_3)}{z-s_1} + \frac{\sigma_1\sigma_2(s_2-s_3)(s_2-s_1)}{z-s_2} + \frac{\tau_1\tau_2(s_3-s_1)(s_3-s_2)}{z-s_3}\right)$$

$$\times \frac{f(z)}{\prod_{j=1}^{3}(z-s_j)} = 0,$$

where $\rho_1, \rho_2; \sigma_1, \sigma_2; \tau_1\tau_2$ are the exponets of (7) respectively at the singular points s_1, s_2, s_3 and satisfy Fuchs's relation (3):

$$\rho_1 + \rho_2 + \sigma_1 + \sigma_2 + \tau_1 + \tau_2 = 1.$$

All solutions of Riemann equations are denoted by the symbol

$$f = P\left\{\begin{matrix} s_1 & s_2 & s_3 & \\ \rho_1 & \sigma_1 & \tau_1 & z \\ \rho_2 & \sigma_2 & \tau_2 & \end{matrix}\right\}.$$

The exponents of the equation are invariants of the conformal transformation

$$w = R(z) = \frac{az+b}{cz+d}$$

of the Riemann sphere and if $w_j = R(s_j)$, then

$$P\left\{\begin{matrix} s_1 & s_2 & s_3 & \\ \rho_1 & \sigma_1 & \tau_1 & z \\ \rho_2 & \sigma_2 & \tau_2 & \end{matrix}\right\} = P\left\{\begin{matrix} w_1 & w_2 & w_3 & \\ \rho_1 & \sigma_1 & \tau_1 & w \\ \rho_2 & \sigma_2 & \tau_2 & \end{matrix}\right\}.$$

If $R(s_1) = 0, R(s_2) = 1,\ R(s_3) = \infty$, then

$$P\left\{\begin{matrix} s_1 & s_2 & s_3 & \\ \rho_1 & \sigma_1 & \tau_1 & z \\ \rho_2 & \sigma_2 & \tau_2 & \end{matrix}\right\} = P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\} = F(\alpha, \beta, \gamma; z).$$

and this is a complete set of solutions of the Gauss hypergeometric equation

$$z(z-1)f^{''}(z) + (\gamma - (\alpha+\beta+1)z)f^{'}(z) - \alpha\beta f(z) = 0. \tag{8}$$

It is known that (8) is *free from accessory parameters* and the monodromy representation of this equation is determined by exponents up to conjugation in $GL_2(\mathbb{C})$.

Consider a second order equations of Fuchsian type whose singularities and exponents are given by the following Riemann scheme

$$P\left\{\begin{array}{cccccc} 0 & 1 & \alpha_1 & ... & \alpha_k & \infty \\ 0 & 0 & 0 & ... & 0 & \alpha & z \\ 1-\gamma & 1-\delta & 1-\epsilon_1 & ... & 1-\epsilon_k & \beta \end{array}\right\}$$

where the exponents are connected by Fuch's relation (see [4], [12], [13])

$$\alpha+\beta-\gamma-\delta-\epsilon_1-...-\epsilon_k+1=0.$$

The equation defined by these date contains just k accessary parameters, denoted by $\rho_1, ..., \rho_k$ and can be written in the form

$$y^{''}(z)+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\sum_{i=1}^{k}\frac{\epsilon_i}{z-a_i}\right)y^{'}(z)+\frac{\alpha\beta z^k+\rho_1 z^{k-1}+...+\rho_k}{z(z-1)\prod_{i=1}^{k}(z-a_i)}=0. \quad (9)$$

In case when $k=1$ we obtain *Heun's* equation. The special type Heun's equation [4], [13] with the Riemann scheme

$$P\left\{\begin{array}{ccccc} 0 & 1 & s & \infty \\ 0 & 0 & 0 & -n/2 & z \\ 1/2 & 1/2 & 1/2 & \frac{1}{2}(n+1) \end{array}\right\},$$

which has the form

$$y^{''}(z)+\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-s}\right)y^{'}(z)+\frac{\lambda-n(n+1)z}{4z(z-1)(z-s)}y(z)=0, \quad (10)$$

is a *Lame* equation, where λ is a accessory parameter.

If the monodromy representation of equation (9) is irreducible and singular points $a_1, ..., a_k$ are apparent ([6], [13], [14]) then the monodromy group with respect to a suitable fundamental system of solutions can be calculated in an explicit form by algebraic operations.

2. The Fuchsian systems with three singular points

Introduce the following differential operators: $D=\frac{d}{dz}$, $\delta=zD$ and $\delta^2=z^2D^2+zD$. In this notations the hypergeometric equation with parameters α, β, γ can be written in following equivalent form:

$$(1-z)\delta^2 F+((\gamma-1)-(\alpha+\beta)z)\delta F-\alpha\beta z F=0, \quad (11)$$

$$z(1-z)D^2F+((\gamma-(\alpha+\beta+1)z)DF-\alpha\beta F=0. \quad (12)$$

The indicial equation at the point 0 of equation (11) is

$$\lambda^2+(\gamma-1)\lambda=0$$

and therefore the exponents are $0, 1-\gamma$. The equation is nonresonant since $\gamma\notin\mathbb{Z}$.

A basis of the solutions near the singular point 0 is:

$$\mathcal{B}_0=\left\{F(\alpha,\beta,\gamma;z), z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z)\right\}. \quad (13)$$

To obtain solutions at the point 1 we change a local coordinate at $z = 1$ take $\tau = 1 - z$. From this if $F(z) = G(\tau)$, then $F^{'}(z) = -G^{'}(\tau)$ and $F^{''}(z) = G^{''}(\tau)$. The equation (11) for $G(\tau)$ has the form

$$\tau(1-\tau)D^2G + ((\alpha+\beta-\gamma+1) - (\alpha+\beta+1)\tau)DG - \alpha\beta G = 0. \tag{14}$$

The solutions of (13) are hypergeometric functions with parameters $\alpha, \beta, \alpha + \beta - \gamma + 1$ and basis of solutions space at 1 are the following functions

$$F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \tag{15}$$

$$(1-z)^{\gamma-\alpha-\beta}F(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1; 1-z). \tag{16}$$

Similarly, not difficult but deep analysis show that change of variable $w = \frac{1}{z}$ gives the following hypergeometric equation with parameters $\alpha, \alpha-\gamma+1, \alpha-\beta+1$:

$$w(1-w)\delta^2H + ((\alpha-\beta) - (2\alpha-\gamma+1)w)\delta H - \alpha(\alpha-\gamma+1)wH = 0. \tag{17}$$

The basis of solutions space of the equation (17) at singular point ∞ is

$$\mathcal{B}_\infty = \left\{ z^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1; 1/z), z^{-\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1; 1/z) \right\}. \tag{18}$$

Note that the hypergeometric function $F(\alpha, \beta, \gamma; z)$ is defined by the series

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n$$

for $|z| < 1$ and by continuation elsewhere [1]. For $\delta \in \mathbb{C}$ and $n \in \mathbb{N}$ by definition $(\delta)_n$ is

$$(\delta)_0 = 1 \ \text{ and } \ (\delta)_n = \delta(\delta+1)...(\delta+n-1), \quad n \geq 1.$$

We also require that $\alpha, \beta, \gamma \notin -\mathbb{N}$ and $\gamma, \alpha-\beta, \gamma-\alpha-\beta \notin \mathbb{Z}$.

Let

$$\mathcal{S} = \mathbb{CP}^1 \setminus ([\infty, 0] \cup [1, \infty]) = \mathbb{C} \setminus ([-\infty, 0] \cup [1, \infty]).$$

The introduced set $\mathcal{S}$ is simply connected and all three bases of the germs $\mathcal{B}_{\prime}, \mathcal{B}_\infty, \mathcal{B}_\infty$ extend to the bases of the solution space $\mathcal{V}_{\mathcal{S}}(\alpha, \beta, \gamma)$ of the equation (11) (or (12)). For the multivalued functions (such ar z^δ or $(1-z)^\delta$) we always use the principal branch.

Denote by $\pi_1(X_3, z_0)$ the fundamental group of the noncompact Riemann surface $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$, where z_0 is some point from X_3, say $z_0 = 1/2$. Let g_0, g_1 and g_∞ be the loops which begin and end at point z_0 and going around $0, 1, \infty$ respectively. The loops g_0, g_1, g_∞ are generators of $\pi_1(X_3, z_0)$ and satisfy the relation $g_0g_1 = g_\infty^{-1}$.

The equation (11) defines the monodromy representation

$$\rho : \pi_1(X_3, z_0) \to GL_2(\mathbb{C}), \quad \rho(g_j) = M_j, \quad j = 0, 1, \infty. \tag{19}$$

The monodromy matrices M_j satisfy the relation $M_1M_2 = M_\infty^{-1}$. Monodromy group of the equation (11) is generated by the monodromy matrices M_1 and M_2.

Proposition 1. *The monodromy matrices along the loops g_0, g_1, g_∞ with respect to the basis $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_\infty$ are*

$$M_0^{g_0}(\mathcal{B}_0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i\gamma} \end{pmatrix}, \quad M_1^{g_1}(\mathcal{B}_1) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i(\gamma-\alpha-\beta)} \end{pmatrix}$$

$$M_\infty^{g_\infty}(\mathcal{B}_\infty) = \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix},$$

respectively.

The matrices $M_0^{g_0}(\mathcal{B}_0), M_1^{g_1}(\mathcal{B}_1), M_\infty^{g_\infty}(\mathcal{B}_\infty)$ are called local monodromies.

General solutions with respect to bases $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_\infty$ are (and therefore the generic elements of the solutions space $\mathcal{V}_\mathcal{S}(\alpha,\beta,\gamma)$)

$$c_1F(\alpha,\beta,\gamma;z) + c_2z^{1-\gamma}F(\alpha,\beta,\gamma;z),$$

$$c_3F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z)+c_4(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z),$$

$$c_5(-z)^{-\alpha}F(\alpha,\alpha+1-\gamma,\alpha-\beta+1;\frac{1}{z}) + c_6(-z)^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1;\frac{1}{z}),$$

respectively, where $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{C}$ are some constants.

For example, if

$$c_1 = \frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\gamma-1)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}, \quad c_2 = \frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(\beta)},$$

$$c_3 = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad c_4 = \frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)},$$

$$c_5 = \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}, \quad c_6 = \frac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)},$$

then

$$c_1F(\alpha,\beta,\gamma;z) + c_2z^{1-\gamma}F(\alpha,\beta,\gamma;z) = F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z) \in \mathcal{B}_1,$$

$$c_3F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z)+c_4(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z)$$
$$= F(\alpha,\beta,\gamma;z) \in \mathcal{B}_0,$$

$$c_5(-z)^{-\alpha}F(\alpha,\alpha+1-\gamma,\alpha-\beta+1;\frac{1}{z}) + c_6(-z)^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1;\frac{1}{z})$$
$$= F(\alpha,\beta,\gamma;z) \in \mathcal{B}_0.$$

From this we get

Theorem 1. [3], [13] *1. There exists the linear isomorphism*

$$R : \mathcal{V}_\mathcal{S}(\alpha,\beta,\gamma) \to \mathcal{V}_\mathcal{S}(\alpha,\beta,\gamma)$$

such that $R\mathcal{B}_0 = \mathcal{B}_\infty$, where

$$R = \begin{pmatrix} e^{-i\pi\alpha}\dfrac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} & e^{-i\pi(\alpha-\gamma+1)}\dfrac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\Gamma(2-\alpha-1)} \\ e^{-i\pi\beta}\dfrac{\Gamma(\alpha-\beta)\Gamma(\alpha)}{\Gamma(\gamma-\beta)\Gamma(\beta)} & e^{-i\pi(\beta-\gamma+1)}\dfrac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma+1)\Gamma(2-\beta-1)} \end{pmatrix}.$$

2. The monodromy group respect to basis $\mathcal{B}_I$ generated by $M_0^{g_0}(\mathcal{B}_0)$ and $M_0^{g_\infty}(\mathcal{B}_0) = R^{-1}M_\infty^{g_\infty}(\mathcal{B}_\infty)R$.

For any fundamental system of solutions (φ,ψ) of the equation (8) we obtain two monodromy matrices M_0 and M_1 with respect to (φ,ψ). As mentioned above, a change of fundamental system of solutions of (8) induces a conjugate of monodromy representations (and vice versa.) The eigenvalue of M_0 and M_1 are $(1,\exp(2\pi i(1-\gamma)))$ and $(1,\exp(2\pi i(\gamma-\alpha-\beta)))$, respectively.

From (6) it follows that the conjugace classes of the monodromy representation of the equation (8) generated by M_0, M_1 are parametrized by three complex parameters.

Proposition 2. (see [3], [13]) 1. The monodromy group of the hypergeometric equation (8) is irreducible if and only if none of the numbers $\alpha,\beta,\gamma-\alpha,\gamma-\beta$ is an integer.

2. The conjugate classes of the irreducible monodromy representation of equation (8) are generated by matrices of the form

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & \xi \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & \zeta \\ 0 & \eta \end{pmatrix}, \quad \xi \neq 0, \eta \neq 0, \zeta \neq 0, -(\xi-1)(\eta-1), \tag{20}$$

$$M_0 = \begin{pmatrix} \xi & 0 \\ 1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} \eta & \zeta \\ 0 & 1 \end{pmatrix}, \quad \xi \neq 0, \eta \neq 0, \zeta \neq 0, -(\xi-1)(\eta-1), \tag{21}$$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & \xi \end{pmatrix}, M_1 = \begin{pmatrix} \eta & \zeta+(\xi-1)(\eta-1) \\ 0 & 1 \end{pmatrix}, \xi \neq 0, \eta \neq 0, \zeta \neq 0, (\xi-1)(\eta-1), \tag{22}$$

$$M_0 = \begin{pmatrix} \xi & 0 \\ 1 & 1 \end{pmatrix}, M_1 = \begin{pmatrix} 1 & \zeta+(\xi-1)(\eta-1) \\ 0 & \eta \end{pmatrix}, \xi \neq 0, \eta \neq 0, \zeta \neq 0, (\xi-1)(\eta-1), \tag{23}$$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, \quad M_1 = \begin{pmatrix} \eta & 0 \\ 1 & 1 \end{pmatrix}, \tag{24}$$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & \eta \end{pmatrix}, \tag{25}$$

$$M_0 = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} \eta & 0 \\ 1 & 1 \end{pmatrix}, \tag{26}$$

where ξ,η,ζ are expressed by α,β,γ in an algebraic form.

3. Examples

1. First consider the scalar differential equation, i.e. A_i be real or complex numbers. Then $T_1 = (z-a_1)^D$, where $D=1$ or $D=-1$. We obtain

$$\frac{dy_1}{dz} = \left(\sum_{j=1}^{m} \frac{T_1(z)B_1T_1^{-1}(z)}{z-a_1} + \frac{dT_1(z)}{dz}T_1^{-1}(z)\right), y_1 = \left(\frac{B_1+D}{z-a_1} + \sum_{j=2}^{m}\frac{B_j}{z-a_j}\right) y_1$$

and if $\lambda = (\lambda_1, ..., \lambda_m)$, then the transform function will be

$$T(z) = (z-a_1)^{\pm|[Re(B_1)]-\lambda_1|} ... (z-a_m)^{\pm|[Re(B_m)]-\lambda_m|}.$$

2. (Hypergeometric system) Consider the second order system

$$\frac{dy}{dz}=\left(\frac{\begin{pmatrix}0 & 0\\ 0 & \alpha+\beta+1\end{pmatrix}}{z}+\frac{\begin{pmatrix}0 & 0\\ \alpha\beta & -\gamma\end{pmatrix}}{z+2}+\frac{\begin{pmatrix}0 & 0\\ -\alpha\beta & -(\alpha+\beta+1)+\gamma\end{pmatrix}}{z-2}\right)y. \tag{27}$$

i) Let $\alpha=0$, $\beta=-2$ and $\gamma=-2$ in (27). The Transform function for canonical extension has the form (see [11], [14])

$$T(z)=\begin{pmatrix}1 & 0\\ 0 & \frac{z(z-2)}{(z+2)^2}\end{pmatrix}.$$

Factorization of this matrix function is (see [1], [2], [8])

$$T(t)=\begin{pmatrix}1 & 0\\ 0 & \frac{t(t-2)}{(t+2)^2}\end{pmatrix}=\frac{1}{(t+2)^2}\begin{pmatrix}3 & 3t-7\\ 0 & 3\end{pmatrix}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Hence $K=(1,0)$.

ii) Suppose $\alpha=0$, $\beta=-2$ and $\gamma=-2$ in (27) as above and consider the system

$$\frac{dy}{dz}=\left(\frac{\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}}{z}+\frac{\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}}{z+1/3}+\frac{\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}}{z+1/2}\right)y.$$

The transform function, corresponding to the canonical extension (see [5], [6], [9], [10]), is

$$T(z)=\begin{pmatrix}1 & 0\\ 0 & \frac{z(z+1/2)}{(z+1/3)^2}\end{pmatrix}=(z+1/3)^{-2}T_1(z),$$

where $T(z)$ ia a polynomial matrix function

$$T_1(z)=\begin{pmatrix}(z+1/3)^2 & 0\\ 0 & (z+1/3)^2z(z+1/2)\end{pmatrix}.$$

Factorization of $T_1(z)$ on the unit circle is (see [1], [2], [8])

$$T_1(z)=\begin{pmatrix}t^4 & 0\\ 0 & t^2\end{pmatrix}\begin{pmatrix}0 & 1+\frac{7}{6t}+\frac{4}{9t^2}+\frac{1}{18t^3}\\ 1+\frac{2}{3t}+\frac{1}{6t^2} & 0\end{pmatrix}.$$

Therefore $K=(2,0)$.

iii) Let $\alpha = 0$, $\beta = -2$, $\gamma = -2$ in (27) and let $\Lambda = ((1,0),(0,1),(1,1))$. Consider the system

$$\frac{dy}{dz} = \left(\frac{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}{z} + \frac{\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}}{z+1/3} + \frac{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{z+1/2} \right) y. \tag{28}$$

Then

$$T(z) = \begin{pmatrix} \frac{z}{4}(3z-7)(z+1/2) & \frac{3}{4}(z+1/3)(z+1/2) \\ \frac{3}{4}z(z+1/2) & 0 \end{pmatrix}.$$

Factorization is

$$T(t) = \begin{pmatrix} \frac{3}{4} & \frac{3}{4}t - 7/4 \\ 0 & 3/4 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} 0 & 1 + \frac{5}{6t} + \frac{1}{6t^2} \\ 1 + \frac{1}{2z} & 0 \end{pmatrix}.$$

Hence $K = (2,2)$.

iv) Consider system (28) and suppose $\Lambda = ((1,0),(0,1),(-1,-1))$. Then

$$T(z) = \begin{pmatrix} \frac{z(3z-7)}{4(z+1/2)} & \frac{3(z+1/3)}{4(z+1/2)} \\ \frac{3z}{4(z+1/2)} & 0 \end{pmatrix}.$$

$$T(z) = (z+2)^{-2} T_1(z),$$

where

$$T_1(z) = \begin{pmatrix} (z+2)^2 & 0 \\ 0 & z(z+2)^2(z-2) \end{pmatrix}.$$

Factorization $T_1(z)$ has the form

$$T_1(t) = \begin{pmatrix} \frac{3}{4} & \frac{3}{4}t - 7/4 \\ 0 & 3/4 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & 1 + \frac{1}{3t} \\ 1 & 0 \end{pmatrix},$$

From this it follows, that $K = (0,0)$.

3. Consider the following Fuchsian system with singular points $0, \pm 1$

$$\frac{dy}{dz} = \left(\frac{\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}}{z} + \frac{\begin{pmatrix} -1/4 & 1/2 \\ 1/8 & 1/4 \end{pmatrix}}{z+1} + \frac{\begin{pmatrix} 1/4 & -1/2 \\ 1/8 & 3/4 \end{pmatrix}}{z-1} \right) y.$$

The transform function for the canonical extension is

$$T(z) = \begin{pmatrix} -2 & 4z \\ 1 & 2z \end{pmatrix},$$

factorization of $T(z)$ on the unit circle is

$$T(t) = \begin{pmatrix} -2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix},$$

where $|t| = 1$. Hence $K = (0, 1)$.

4. (Example 1 from [14]) Consider the following system of Fuchsian differential equations with singular points $0, 1, 2$

$$\frac{dy}{dz} = \left(\frac{\begin{pmatrix} -3/2 & 0 \\ 1/2 & -3/2 \end{pmatrix}}{z-2} + \frac{\begin{pmatrix} 1 & 0 \\ -3/4 & 2 \end{pmatrix}}{z} + \frac{\begin{pmatrix} 1/2 & 0 \\ 1/4 & -1/2 \end{pmatrix}}{z-1} \right) y.$$

The transform function for the canonical extension has the form

$$T(z) = \begin{pmatrix} \frac{(z-2)^2(z-1)(2z-3)}{4z^2} & \frac{(z-2)^2(z-1)}{z^2} \\ \frac{(z-2)^2}{2z} & 0 \end{pmatrix}.$$

Factorizing this matrix function on $|t| = 1$ we obtain

$$T(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \frac{-(t-2)^2(5t-3)}{4t^3} & \frac{(t-2)^2(t-1)}{t^3} \\ \frac{(t-2)^2}{2t^2} & 0 \end{pmatrix}.$$

From this the splitting type of canonical extension is $K = (1, 1)$.

5. Consider second order Fuchsian system with four singular points and valuations $(1, -1)$, $(0, 0)$, $(0, 0)$, $(0, 0)$ at the points 0, -1, 1, $1/2$, respectively:

$$\frac{dy}{dz} = \left(\frac{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{z} + \frac{\begin{pmatrix} -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} \end{pmatrix}}{z+1} + \frac{\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}{z-1} + \frac{\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}}{z-1/2} \right) y.$$

The transform matrix corresponding to valuations $\Lambda = ((0, 0), (0, 0), (0, 0), (0, 0))$ is

$$T(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}.$$

Hence $K = (1, -1)$.

6. Consider the following nonresonant system

$$\frac{dy}{dz} = \left(\frac{\begin{pmatrix} -\frac{12}{8} & 0 \\ \frac{1}{4} & -\frac{10}{8} \end{pmatrix}}{z} + \frac{\begin{pmatrix} \frac{11}{8} & 0 \\ -\frac{1}{4} & \frac{12}{8} \end{pmatrix}}{z-1/2} + \frac{\begin{pmatrix} \frac{1}{8} & 0 \\ 0 & -\frac{2}{8} \end{pmatrix}}{z-i/2} \right) y.$$

with singular points $0, 1/2, i/2$. Transform function for canonical extension has the form

$$T(z) = \begin{pmatrix} \frac{z^2}{z-1/2} & 0 \\ 0 & \frac{z^2(z-i/2)}{z-1/2} \end{pmatrix}$$

$$T(z) = (z-1/2)^{-1} T_1(z),$$

where

$$T_1(z) = \begin{pmatrix} z^2 & 0 \\ 0 & z^2(z-i/2) \end{pmatrix}.$$

Factorization of $T_1(z)$ is

$$T_1(t) = \begin{pmatrix} \frac{3}{4} & \frac{3}{4}t - 7/4 \\ 0 & 3/4 \end{pmatrix} \begin{pmatrix} t^3 & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} 0 & 1 - \frac{i/2}{t} \\ 1 & 0 \end{pmatrix},$$

$ind(z-1/2) = 1$, therefore $K = (2, 1)$.

The local monodromy matrices corresponding to singular points $0, 1/2, i/2$ are

$$M_1(t) = \begin{pmatrix} -1 & 0 \\ 1-i & i \end{pmatrix}, M_2(t) = \begin{pmatrix} \frac{\sqrt{2}+i\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}+i\sqrt{2}+2}{11} & -1 \end{pmatrix}, M_3(t) = \begin{pmatrix} \frac{\sqrt{2}+i\sqrt{2}}{2} & 0 \\ 0 & -i \end{pmatrix}$$

respectively.

Consider the triangle with vertices at points $(0,0), (1/2,0)$ and $(1/2,0)$ as a closed contour on the complex plane. Let $f(t)$ piecewise constant matrix function be given as: $f(t) = M_1$ if $t \in (0, 1/2]$; $f(t) = M_2 M_1$, if $t \in (1/2, i/2]$ and $f(t) = M_3 M_2 M_1$, if $t \in (i/2, 0]$. Then partial indices of the Riemann-Hilbert boundary value problem

$$\varphi^+(t) = f(t)\varphi^-(t)$$

equal to $(2, 1)$.

7. The second order Fuchsian system with four singular points

$$\frac{dy}{dz} = \left(\frac{A_1}{z + \frac{1}{2} - \frac{3i}{2}} + \frac{A_2}{z - \frac{1}{3} - \frac{3i}{2}} + \frac{A_3}{z - \frac{1}{2} + \frac{i}{4}} + \frac{A_4}{z - \frac{1}{3} + \frac{i}{3}} \right) y.$$

$$A_1 = \begin{pmatrix} -\frac{7}{4} & i \\ 0 & -\frac{1}{2} \end{pmatrix}, A_2 = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ 0 & -\frac{1}{3} \end{pmatrix}, A_3 = \begin{pmatrix} \frac{5}{4} & -i \\ 0 & -\frac{1}{3} \end{pmatrix}, A_4 = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{4} \end{pmatrix}.$$

The transform matrix for valuations $\Lambda = ((0,0), (0,0), (0,0), (0,0))$ (corresponding to canonical extension of vector bundle or system of standard form) is

$$T(z) = \frac{1}{z - 1/2 + i/4} \begin{pmatrix} T_{11} & T_{12} \\ 0 & -T_{22} \end{pmatrix},$$

where

$$T_{11} = (z + 1/2 - 3i/4)^2 (z - 1/3 - 3i/2);$$

$$T_{12} = (-12/11 - 12i/11)(z + 1/2 - 3i/4)(z - 1/3 - 3i/2);$$
$$T_{22} = (z - 1/2 + i/4)(z + 1/2 - 3i/4)(z - 1/3 - 3i/2).$$

The partial indices of $T(z)$ and therefore splitting type of associated vector bundle are $k_1 = 1$, $k_2 = 1$.

8. Consider the following system of Fuchsian differential equations with five singular points $0, \pm 1, \mp i$ (example 3 from [13]):

$$\frac{dy}{dz} = \left(\frac{A_1}{z} + \frac{A_2}{z+1} + \frac{A_3}{z-1} + \frac{A_4}{z+i} + \frac{A_5}{z-i} \right),$$

where

$$A_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1/8 & -1/16 \\ 1/16 & -1/4 \end{pmatrix}, A_3 = \begin{pmatrix} -1/8 & -1/16 \\ 1/16 & -1/4 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -1/8 & 1/16 \\ -1/16 & -1/4 \end{pmatrix}, A_5 = \begin{pmatrix} -1/8 & 1/16 \\ -1/16 & -1/4 \end{pmatrix},$$

or more shortly

$$\frac{dy}{dz} = \left(\frac{\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}}{2z} + \frac{\begin{pmatrix} -2 & -1 \\ 1 & 4 \end{pmatrix} z}{8(z^2-1)} + \frac{\begin{pmatrix} -2 & 1 \\ -1 & 4 \end{pmatrix} z}{8(z^2+1)} \right) y.$$

Suppose $\Lambda = ((1,0), (-1,-1), (-1,-1), (-1,-1), (-1,-1))$, then the transform function has the form

$$T(z) = \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}.$$

After the right factorization of $T(z)$ we obtain

$$T(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and therefore the splitting type of associated holomorphic vector bundle is $(-1, 1)$.

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R E F E R E N C E S

1. Adukov V.M. Wiener-Hopf factorization of piecewise meromorphic matrix-valued functions. *Sb. Math.*, **200**, 8 (2009), 1105-1126.

2. Adukov V.M. Algorithm of polynomial factorization and its implementation in Maple. *Vestnik SUSU*, **11**, 4 (2018), 110-122.

3. Andrews G., Askey R., Roy R. Spetial Functions. *Cambridge Uni. Press*, 2000.

4. Anosov D., Bolibruch A. The Riemann-Hilbert Problem. *Vieweg+Teubner Verlag*, 1994.

5. Bojarski B., Giorgadze G. Some analytical and geometric aspects of the stable partial indices. *Proceedings of I. Vekua Institute of Applied Mathematics*, **61-62** (2011-2012), 14-32.

6. Bolibrukh A. A. On sufficient conditions for the existence of a Fuchsian equation with prescribed monodromy. *J. Dynam. Control Systems*, **1**, 2 (1995), 329-352.

7. Coddington E., Levinson N. Theory of Ordinary Differential Equations. *McGraw-Hill*, 1987.

8. Ehrhardt T., Spitkovsky I.M. Factorization of piecewise constant matrix functions and systems of linear differential equations. *St. Petersburg Math. J.*, **13**, 6 (2002), 939-991.

9. Giorgadze G. *G*-systems and holomorphic principal bundles on Riemann surfaces. *J. Dyn. Contr. Syst.*, **8**, 2 (2002), 245-291.

10. Giorgadze G. Regular systems on Riemann surfaces. *J. Math. Sci. (N.Y)*, **118** (2003), 5347-5399.

11. Gulagashvili G. On the splitting type of vector bundle on Riemann sphere. *Proceedings of I. Vekua Institute of Applied Mathematics*, **68** (2018), 36-45.

12. Hartman Ph. Ordinary Differential Equations. *SIAM*, 2002.

13. Iwasaki K., Kimura H., Shimomura S., Yoshida M. From Gauss to Painlev: A Modern Theory of Special Functions. *Springer*, 1991.

14. Ryabov A.A. Splitting types of vector bundles constructed from the monodromy of a given Fuchsian system. *Comput. Math. and Math. Phys.*, **44**, 4 (2004), 640-648.

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