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A Relation Between Monodromy Matrix and Accessory Parameter of Lamé Equation

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We will consider the Lamé equation, which is a second-order differential equation with an accessory parameter. It is known that the existence of an accessory parameter depends on the monodromy matrices. We will determine the type of this dependence.

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1 An accessory parameter of Lamé equation

Consider the Lamé equation:

$$p(z)y^{''} + \frac{1}{2}p^{'}(z)y^{'} - (n\,(n+1)\,z + B)\,y = 0, \tag{1.1}$$

we are given

$$p(z) = 4z^3 - g_2 z - g_3 = 4\prod_{i=1}^{3}(z - z_i) = (z - z_1)(z - z_2)(z - z_3) \tag{1.2}$$

a polynomial of 3rd degree in a complex variable, which is presented in its canonical form (meaning it has no quadratic term). Its roots are $z_1, z_2, z_3 \in \mathbb{C}$ complex numbers. The other known parameters are $n \in \mathbb{Q}$, g_2, g_3, and B. The parameter B is the accessory parameter of Lamé equation.

If we rewrite equation (1) for its highest-order derivative, we get the following expression:

$$y^{''} = -\frac{1}{2}\frac{p^{'}(z)}{p(z)}y^{'} + \frac{n(n+1)z + B}{p(z)}y. \tag{1.3}$$

If we introduce the substitutions $x_1 = y$, $x_2 = y^{'}$ and define the column vector X=$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then equation (3) is transformed into a system of equations, which can be given in matrix form as

$$\begin{cases} x_1^{'} = x_2 \\ x_2^{'} = -\frac{1}{2}\frac{p^{'}(z)}{p(z)}x_2 + \frac{n(n+1)z + B}{p(z)}x_1 = \frac{n(n+1)z + B}{p(z)}x_1 - \frac{1}{2}\frac{p^{'}(z)}{p(z)}x_2 \end{cases} \tag{1.4}$$

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In a matrix form, this is given as:

$$\frac{dX}{dz} = \begin{pmatrix} 0 & 1 \\ \dfrac{n(n+1)z+B}{p(z)} & -\dfrac{1}{2}\dfrac{p^{'}(z)}{p(z)} \end{pmatrix} X, \tag{1.5}$$

where

$$B(z) = \begin{pmatrix} 0 & 1 \\ \dfrac{n(n+1)z+B}{p(z)} & -\dfrac{1}{2}\dfrac{p^{'}(z)}{p(z)} \end{pmatrix} \tag{1.6}$$

is the matrix of system (5), which is equivalent to the following Fuchsian system:

$$\frac{dF}{dz} = \left(\frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_3}{z-z_3} \right) F, \tag{1.7}$$

where $A_1 = \underset{z_1}{res} B(z)$, $A_2 = \underset{z_2}{res} B(z)$, $A_3 = \underset{z_3}{res} B(z)$. To calculate these residues, we first need to simplify the terms of the $B(z)$ matrix. Let's try to do this. First, we will calculate the derivative of the polynomial $p(z)$,

$$p^{'}(z) = 4\left((z-z_1)(z-z_2) + (z-z_1)(z-z_3) + (z-z_2)(z-z_3) \right) \tag{1.8}$$

which is needed to simplify the expression:

$$\begin{aligned} \frac{p^{'}(z)}{p(z)} &= \frac{4\left((z-z_1)(z-z_2) + (z-z_1)(z-z_3) + (z-z_2)(z-z_3) \right)}{4(z-z_1)(z-z_2)(z-z_3)} \\ &= \frac{1}{z-z_1} + \frac{1}{z-z_2} + \frac{1}{z-z_3}. \end{aligned} \tag{1.9}$$

To calculate the second entry of the $B(z)$ matrix, we first need to simplify the following expression:

$$\begin{aligned} &\frac{1}{(z-z_1)(z-z_2)(z-z_3)} = \frac{1}{(z-z_1)(z-z_2)} \cdot \frac{1}{z-z_3} \\ &= \frac{1}{(z_1-z_2)(z_1-z_3)} \cdot \frac{1}{z-z_1} + \frac{1}{(z_2-z_1)(z_2-z_3)} \cdot \frac{1}{z-z_2} \\ &+ \frac{1}{(z_3-z_1)(z_3-z_2)} \cdot \frac{1}{z-z_3}, \end{aligned} \tag{1.10}$$

and then simplify a second one:

$$\frac{\alpha \cdot z + \beta}{z-\gamma} = \frac{\alpha \cdot (z-\gamma) + \beta + \alpha \cdot \gamma}{z-\gamma} = \alpha + \frac{\beta + \alpha \cdot \gamma}{z-\gamma}. \tag{1.11}$$

Finally, we are ready to simplify the second entry of $B(z)$, which is the following expression:

$$\begin{aligned}
&4\cdot\frac{n(n+1)z+B}{p(z)}=4\cdot(n(n+1)z+B)\cdot\frac{1}{p(z)}\\
&=\frac{1}{(z_1-z_2)(z_1-z_3)}\cdot\frac{n(n+1)z+B}{z-z_1}+\frac{1}{(z_2-z_1)(z_2-z_3)}\cdot\frac{n(n+1)z+B}{z-z_2}\\
&+\frac{1}{(z_3-z_1)(z_3-z_2)}\cdot\frac{n(n+1)z+B}{z-z_3}\\
&=\frac{B+n(n+1)z_1}{(z_1-z_2)(z_1-z_3)}\cdot\frac{1}{z-z_1}+\frac{B+n(n+1)z_2}{(z_2-z_1)(z_2-z_3)}\cdot\frac{1}{z-z_2}\\
&+\frac{B+n(n+1)z_3}{(z_3-z_1)(z_3-z_2)}\cdot\frac{1}{z-z_3}.
\end{aligned}\tag{1.12}$$

Considering all of this, we can conclude that:

$$B(z)=\begin{pmatrix}0 & 1\\ \dfrac{n(n+1)z+B}{p(z)} & -\dfrac{1}{2}\dfrac{p'(z)}{p(z)}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 1\\ \begin{aligned}&\frac{1}{4}\cdot\Bigg(\frac{B+n(n+1)z_1}{(z_1-z_2)(z_1-z_3)}\cdot\frac{1}{z-z_1}\\ &+\frac{B+n(n+1)z_2}{(z_2-z_1)(z_2-z_3)}\cdot\frac{1}{z-z_2}\\ &+\frac{B+n(n+1)z_3}{(z_3-z_1)(z_3-z_2)}\cdot\frac{1}{z-z_3}\Bigg)\end{aligned} & -\dfrac{1}{2}\cdot\left(\dfrac{1}{z-z_1}+\dfrac{1}{z-z_2}+\dfrac{1}{z-z_3}\right)\end{pmatrix}$$

$$\begin{aligned}
&=\begin{pmatrix}0 & \dfrac{1}{3}\\ \dfrac{B+n(n+1)z_1}{4(z_1-z_2)(z_1-z_3)}\cdot\dfrac{1}{z-z_1} & -\dfrac{1}{2}\cdot\dfrac{1}{z-z_1}\end{pmatrix}\\
&+\begin{pmatrix}0 & \dfrac{1}{3}\\ \dfrac{B+n(n+1)z_2}{4(z_2-z_1)(z_2-z_3)}\cdot\dfrac{1}{z-z_2} & -\dfrac{1}{2}\cdot\dfrac{1}{z-z_2}\end{pmatrix}\\
&+\begin{pmatrix}0 & \dfrac{1}{3}\\ \dfrac{B+n(n+1)z_3}{4(z_3-z_1)(z_3-z_2)}\cdot\dfrac{1}{z-z_3} & -\dfrac{1}{2}\cdot\dfrac{1}{z-z_3}\end{pmatrix}.
\end{aligned}\tag{1.13}$$

We are now ready to calculate the $A_j=\underset{z_j}{res}B(z)=\frac{1}{2\pi i}\int\limits_{\{z:|z-z_j|=r\}}B(z)dz$ matrices for $j=$

1, 2, 3 and $r>0$ is a positive real number. To do this, we first need to calculate the following integrals:

$$\int\limits_{\{z:|z-z_j|=r\}} \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)}\cdot\frac{1}{z-z_j} & -\frac{1}{2}\cdot\frac{1}{z-z_j} \end{pmatrix}$$

$$= \begin{pmatrix} \int\limits_{\{z:|z-z_j|=r\}} 0dz & \int\limits_{\{z:|z-z_j|=r\}} \frac{1}{3}dz \\ \int\limits_{\{z:|z-z_j|=r\}} \frac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)}\cdot\frac{1}{z-z_j}dz & \int\limits_{\{z:|z-z_j|=r\}} \left(-\frac{1}{2}\cdot\frac{1}{z-z_j}\right)dz \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)}\cdot 2\pi i & -\frac{1}{2}\cdot 2\pi i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)}\cdot 2\pi i & -\pi i \end{pmatrix}, \tag{1.14}$$

and then this:

$$\int\limits_{\{z:|z-z_j|=r\}} \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{B+n(n+1)z_k}{4(z_k-z_j)(z_k-z_p)}\cdot\frac{1}{z-z_k} & -\frac{1}{2}\cdot\frac{1}{z-z_k} \end{pmatrix}$$

$$= \begin{pmatrix} \int\limits_{\{z:|z-z_j|=r\}} 0dz & \int\limits_{\{z:|z-z_j|=r\}} \frac{1}{3}dz \\ \int\limits_{\{z:|z-z_j|=r\}} \frac{B+n(n+1)z_k}{4(z_k-z_j)(z_k-z_p)}\cdot\frac{1}{z-z_k}dz & \int\limits_{\{z:|z-z_j|=r\}} \left(-\frac{1}{2}\cdot\frac{1}{z-z_k}\right)dz \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.15}$$

where $j,\ k,\ p=1,\ 2,\ 3$ and $j\neq k\neq p$. According to the definition, we obtain:

$$A_j = \underset{z_j}{res}B(z) = \frac{1}{2\pi i}\int\limits_{\{z:|z-z_j|=r\}} B(z)dz = \frac{1}{2\pi i}\begin{pmatrix} 0 & 0 \\ \frac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)}\cdot 2\pi i & -\pi i \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)} & -\frac{1}{2} \end{pmatrix}, \tag{1.16}$$

where $j,\ k,\ p=1,\ 2,\ 3$ and $j\neq k\neq p$. If we consider the following expression, we obtain:

$$\frac{B+n(n+1)z_1}{4\left(z_1-z_2\right)\left(z_1-z_3\right)}+\frac{B+n(n+1)z_2}{4\left(z_2-z_1\right)\left(z_2-z_3\right)}+\frac{B+n(n+1)z_3}{4\left(z_3-z_1\right)\left(z_3-z_2\right)}$$

$$= \frac{B}{4} \cdot \frac{z_2 - z_3 - z_1 + z_3 + z_1 - z_2}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}$$

$$+ \frac{n(n+1)}{4} \cdot \frac{z_1 z_2 - z_1 z_3 - z_1 z_2 + z_2 z_3 + z_1 z_3 - z_2 z_3}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} = 0. \tag{1.17}$$

from this, we can conclude that:

$$A_\infty = \begin{pmatrix} 0 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}. \tag{1.18}$$

2 The dependence between the accessory parameter and the monodromy matrix of the Lamé equation

It is well known that the calculation of monodromy matrices depends on either a resonance or non-resonance condition. A matrix of this type is given by the Taylor series expansion of the matrix exponential:

$$e^{2\pi iA} = E + 2\pi iA + \frac{(2\pi iA)^2}{2!} + \frac{(2\pi iA)^3}{3!} + \ldots + \frac{(2\pi iA)^n}{n!} + \ldots \tag{2.1}$$

Here, E denotes identity matrix, A denotes the matrix corresponding to a singular point of the Fuchsian system of differential equations. We need to calculate the matrix $e^{2\pi iA}$ for the Fuchsian system induced from the Lamé equation. Let us try to do this.
For this, it is necessary to calculate the following matrices, as we already know the form of the matrix for the Fuchsian system corresponding to the singular points induced from the Lamé equation:

$$e^{2\pi iA} = \begin{pmatrix} 1 & 0 \\ \frac{2\pi ia\left(e^{2\pi ib} - 1\right)}{2\pi ib} & e^{2\pi ib} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{a\left(e^{2\pi ib} - 1\right)}{b} & e^{2\pi ib} \end{pmatrix}. \tag{2.2}$$

Theorem 2.1. *The monodromy matrices at the finite singular points z_j and the infinite singular point ∞ will be calculated using the following formulas accordingly:*

$$e^{2\pi iA_j} = \begin{pmatrix} 1 & 0 \\ \frac{B + n(n+1)z_j}{(z_j - z_k)(z_j - z_p)} & -1 \end{pmatrix},$$

and

$$e^{2\pi iA_\infty} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.3}$$

where $j,\ k,\ p = 1,\ 2,\ 3$ and $j \neq k \neq p$.

Proof. If we observe, we will see that the characteristic roots at the finite singular points z_j and the infinite singular point ∞ are $\left(0,\ -\frac{1}{2}\right)$ and $\left(0,\ \frac{3}{2}\right)$ accordingly. From this, we can conclude that the Fuchsian systems induced from the Lamé equation are non-resonant at these points. This means that the monodromy matrices for the finite and infinite singular points are calculated using the formula $e^{2\pi i A_j}$, where

$$A_j = \begin{pmatrix} 0 & 0 \\ \dfrac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)} & -\dfrac{1}{2} \end{pmatrix},$$

and

$$A_\infty = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{3}{2} \end{pmatrix}, \tag{2.4}$$

$j,\ k,\ p = 1,\ 2,\ 3$ and $j \neq k \neq p$.
As we have seen above, the form of the A_j matrix matches the form of the A matrix for which we have already calculated $e^{2\pi i A}$ matrix. If we substitute $a = \dfrac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)}$ and $b = -\dfrac{1}{2}$ into the formula given above, we obtain:

$$e^{2\pi i A_j} = \begin{pmatrix} 1 & 0 \\ \dfrac{\dfrac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)}\left(e^{2\pi i\left(-\frac{1}{2}\right)}-1\right)}{-\dfrac{1}{2}} & e^{2\pi i\left(-\frac{1}{2}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -2\left(e^{-\pi i}-1\right)\dfrac{B+n(n+1)z_j}{4(z_j-z_k)(z_j-z_p)} & e^{-\pi i} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \dfrac{B+n(n+1)z_j}{(z_j-z_k)(z_j-z_p)} & -1 \end{pmatrix}, \tag{2.5}$$

where $j,\ k,\ p = 1,\ 2,\ 3$ and $j \neq k \neq p$.
If we now substitute $a = 0$ and $b = \dfrac{3}{2}$ into the formula given above, we obtain:

$$e^{2\pi i A_\infty} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\left(\frac{3}{2}\right)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{3\pi i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$

From this, we can conclude how fixing the accessory parameter determines the monodromy matrix.
Let's try to figure out how a condition on a monodromy matrix, such as for example unitarity, determines an accessory parameter.
In our case, the matrix M can take one of two forms:

$$M = \begin{pmatrix} 1 & 0 \\ \dfrac{B+n(n+1)z_j}{(z_j-z_k)(z_j-z_p)} & -1 \end{pmatrix},$$

or

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7}$$

where $j,\ k,\ p = 1,\ 2,\ 3$ and $j \neq k \neq p$. In general, the form of the monodromy matrix M is of the type:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.8}$$

As is known, the unitarity condition is equivalent to the isometry condition, which means that $\|x\|_2 \ = \ \|Mx\|_2$.

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}},$$

and

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} \right\| = \left((ax_1 + bx_2)^2 + (cx_1 + dx_2)^2 \right)^{\frac{1}{2}}, \tag{2.9}$$

where $x_i,\ y_i,\ a,\ b,\ c,\ d$ are complex numbers $i = 1,\ 2$. The isometry condition for a general form of the monodromy matrix means that the matrix preserves the length of a vector. This can be expressed by the following equality:

$$x_1^2 + x_2^2 = \left(a^2 + c^2\right) x_1^2 + 2\left(ab + cd\right) x_1 x_2 + \left(b^2 + d^2\right) x_2^2,$$

and then

$$x_1^2 \left(1 - a^2\right) + x_2^2 \left(1 - d^2\right) = c^2 x_1^2 + 2\left(ab + cd\right) x_1 x_2 + b^2 x_2^2. \tag{2.10}$$

In our case, if we make the substitution $a = 1,\ b = 0,\ d = -1$ we get:

$$x_1^2 \left(1 - 1\right) + x_2^2 \left(1 - 1\right) = c^2 x_1^2 - 2c x_1 x_2 + 0 x_2^2,$$

$$c\left(c x_1 - 2x_2\right) = 0,$$

$$\left[\begin{array}{l} c = 0, \\ c = 2 \cdot \dfrac{x_2}{x_1}. \end{array} \right. \tag{2.11}$$

Let's take into account that in our case $c = \dfrac{B + n(n+1)z_j}{(z_j - z_k)(z_j - z_p)}$ and denote $2 \cdot \dfrac{x_2}{x_1}$ expresion by λ, then we have:

$$\left[\begin{array}{l} \dfrac{B + n(n+1)z_j}{(z_j - z_k)(z_j - z_p)} = 0, \\ \dfrac{B + n(n+1)z_j}{(z_j - z_k)(z_j - z_p)} = \lambda, \end{array} \right.$$

and

$$\left[\begin{array}{l} B = -n(n+1)z_j, \\ B = \lambda \cdot (z_j - z_k)(z_j - z_p) - n(n+1)z_j, \end{array}\right. \tag{2.12}$$

in general:

$$B = \lambda \cdot (z_j - z_k)(z_j - z_p) - n(n+1)z_j, \tag{2.13}$$

where $\lambda \in \mathbb{C}$ and $j,\ k,\ p = 1,\ 2,\ 3$ with $j \neq k \neq p$. □

Theorem 2.2. *If the monodromy matrix at the finite singular point z_j is unitary, then the accessory parameter will be calculated using the following formula:*

$$B = \lambda \cdot (z_j - z_k)(z_j - z_p) - n(n+1)z_j, \tag{2.14}$$

where $\lambda \in \mathbb{C}$ and $j,\ k,\ p = 1,\ 2,\ 3$ with $j \neq k \neq p$.

Proof. The proof can be seen above.

From this, we can conclude how the condition on the monodromy matrix fixes the accessory parameter and vice versa. □

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