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A Dynamical Problem of Zero Approximation of Hierarchical Models for Incompressible Fluids

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We investigate dynamical problem of zero approximation of hierarchical models for fluids. Applying the Laplace transform technique, we reduce the dynamical problem to the elliptic problem which depends on a complex parameter and prove the corresponding uniqueness and existence results.

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1 Introduction

Let a viscous incompressible fluid occupy a region Ω and let $\Gamma = \partial\Omega$, the surface Γ is divide it into two submanifolds, the so-called Dirichlet part Γ^D and Neumann part Γ^N, $\Gamma = \Gamma^D \cup \Gamma^N$, $\Gamma^D \cap \Gamma^N = \emptyset$. For any subset $\mathcal{M} \subset R^3$, let $Z(\mathcal{M}) := \mathcal{M} \times (0, +\infty)$. Throughout the paper, $n \equiv n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward (with respect to Ω) unit normal vector at the point $x \in \Gamma$. The surfaces Γ, Γ^D, Γ^N are assumed to be Lipschitz manifolds.

We denote by $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ and $p(x,t)$ the velocity vector and pressure, respectively, while

$$e_{kj} \equiv e_{kj}(v) := \frac{1}{2}\Big(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k}\Big)$$

is the strein velocity tensor and

$$\sigma_{kj} \equiv \sigma(v,p) := 2\mu e_{kj}(v) - \delta_{kj} p$$

is the stress tensor in the case of viscous incompressible fluid with the viscosity coefficient μ.

Further, we introduce the stress vector and stress operator as follows

$$\sigma_{kj} n_j = [2\mu e_{kj}(v) - \delta_{kj} p] n_j =: \Big\{T(\partial, n)[v,p]\Big\}_k, \tag{1}$$

$$T(\partial, n) := [T_{kj}(\partial, n)]_{3\times 4}, \;\; T_{kj}(\partial, n) := \mu\Big[n_j \frac{\partial}{\partial x_k} + \delta_{kj}\frac{\partial}{\partial n}\Big], \;\; T_{k4}(\partial, n) := -n_k, \tag{2}$$

where $k, j = 1, 2, 3$. Here and in what follows the summation over Latin subindices is meant from 1 to 3, while over the repeated Greek subindices we assume summation from 1 to 2. It should be noted that $[v,p]$ is a four-dimensional vector or can also be treated as a 4×1 column matrix.

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Let us consider prismatic shells, occupying 3D domain Ω with the projection ω (on the plane $x_3=0$) and the face surfaces

$$x_3=\overset{(+)}{h}(x_1,x_2)\in C^2(\omega)\ \text{ and }\ x_3=\overset{(-)}{h}(x_1,x_2)\in C^2(\omega),\ \ (x_1,x_2)\in\omega.$$

$$2h(x_1,x_2):=\overset{(+)}{h}(x_1,x_2)-\overset{(-)}{h}(x_1,x_2)>0,\ \ (x_1,x_2)\in\omega,$$

is the thickness of the prismatic shell. A part of $\partial\omega$, where the thickness vanishes, i.e., $2h=0$, is said to be a cusped edge. We shall call it a blunt edge, if in the symmetric case (see below) $\partial\Omega$ contains it smoothly, otherwise, i.e., the points of the cusped edge are points of nonsmoothness of $\partial\Omega$, we shall call it a sharp edge. Here we consider the case when $2h(x_1,x_2)=const.$

Remark 1.1. The case $h\neq$ conct will be considered in the forthcomming paper.

Let ω be an open, bounded and simply connected subset of R^2, with a smooth boundary $\ell=\ell^D\cup\ell^N:=\partial\omega$, $\ell^D\cap\ell^N=\emptyset$.

We investigate dynamical problem of the zeroth order approximation of hierarchical models for fluids [1], [2]. Applying the Laplace transform technique, we reduce the dynamical problem to the elliptic problem which depends on a complex parameter and prove the corresponding uniqueness and existence results.

2 Title problem

In the $N=0$ approximation hierarchical models for fluids we have the following governing equations (see [1]-[3])

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta}+\Big[\lambda h\overset{0}{\widetilde{v}}_{\gamma0,\gamma}\Big]_{,\beta}+\Big[\mu h\Big(\overset{0}{\widetilde{v}}_{\alpha0,\beta}+\overset{0}{\widetilde{v}}_{\beta0,\alpha}\Big)\Big]_{,\alpha}+\overset{0}{X}_\beta=\rho h\frac{\partial\overset{0}{\widetilde{v}}_{\beta0}}{\partial t},\ \ \beta=1,2;\tag{3}$$

$$\Big(\mu h\overset{0}{\widetilde{v}}_{30,\alpha}\Big)_{,\alpha}+\overset{0}{X}_3=\rho h\frac{\partial\overset{0}{\widetilde{v}}_{30}}{\partial t},\tag{4}$$

where

$$\overset{0}{\widetilde{v}}_{j0}:=\frac{v_{j0}}{h},\ \ \overset{0}{\widetilde{p}}_0:=\frac{p_0}{h},\tag{5}$$

are so called zeroth weighted moments of the velocity vector components and pressure, correspondingly,

$$(v_{j0},p_0)(x_1,x_2):=\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}(v_j,p)(x_1,x_2,x_3)dx_3,\ \ j=1,2,3.$$

The stress vector components on the upper and lower face surfaces be assumed to be known

$$\overset{0}{X}_i:=Q_{\overset{(+)}{n}i}\sqrt{1+\Big(\overset{(+)}{h}_{,1}\Big)^2+\Big(\overset{(+)}{h}_{,2}\Big)^2}+Q_{\overset{(-)}{n}i}\sqrt{1+\Big(\overset{(-)}{h}_{,1}\Big)^2+\Big(\overset{(-)}{h}_{,2}\Big)^2}+\Phi_{i0},\ \ =\overline{1,3},\tag{6}$$

$$\Phi_{i0}(x_1,x_2):=\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}\Phi_j(x_1,x_2,x_3)dx_3.$$

We naturally use known values of stress vector components $Q_{\underset{n}{(+)}i}$, $Q_{\underset{n}{(-)}i}$ on the face surfaces and the values of Fourier-Legendre expansions of velocities there. Φ_{i0} is the zeroth order moments of the volume forces.

Since we consider incompressible fluid, from the following equation

$$\mathrm{div} v = 0$$

for the weighted moments we have ([1], [3])

$$\overset{0}{\widetilde{v}}_{\gamma 0,\gamma} = 0. \tag{7}$$

From (3), bearing in mind (7), we obtain

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta\beta} = -\overset{0}{X}_{\beta,\beta},.$$

System (3)-(4), (7) may be rewritten as follows

$$(h\overset{0}{\widetilde{p}}_0)_{,i} + \mu h \Delta_2 \overset{0}{\widetilde{v}}_{i0} + \overset{0}{X}_i = \rho h \frac{\partial \overset{0}{\widetilde{v}}_{i0}}{\partial t}, \tag{8}$$

$$\overset{0}{\widetilde{v}}_{\gamma 0,\gamma} = 0, \tag{9}$$

here Δ_2 is a two dimensional Laplace operator.

If $h = const$ we may rewrite (8)-(9) in the following vector form

$$A^f[\overset{0}{\widetilde{v}}] := -\mu^f \, \Delta_2 \, \overset{0}{\widetilde{v}} - \rho \, \frac{\partial \overset{0}{\widetilde{v}}}{\partial t} + \mathrm{grad} \, \overset{0}{\widetilde{p}} = F \ \text{ in } \ Z(\omega), \tag{10}$$

$$\mathrm{div} \overset{0}{\widetilde{v}} = 0 \ \text{ in } \ Z(\omega), \tag{11}$$

where $F := \overset{0}{X}/h$.

For any subset $\mathcal{M} \subset R^2$, let $Z(\mathcal{M}) := \mathcal{M} \times (0, +\infty)$.

Let us denote by L_2, W_2^r and $H_2^s = H^s$ with $r \geq 0$ and $s \in R$ the Lebesgue, SobolevSlobodetski and Bessel potential function spaces, respectively. Recall that $W_2^r = H^r$ for $r \geq 0$.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a smooth proper submanifold $\mathcal{M} \subset \mathcal{M}_0$, we denote by $\widetilde{H}^s(\mathcal{M})$ the subspace of $H^s(\mathcal{M}_0)$,

$$\widetilde{H}^s(\mathcal{M}) := \{g : g \in H^s(\mathcal{M}_0), \ \ \mathrm{supp} g \subset \mathcal{M}\},$$

while $H^s(\mathcal{M})$ stands for the space of restrictions on $\mathcal{M}$ of functions from $H^s(\mathcal{M}_0)$,

$$H^s(\mathcal{M}) := \{r_{\mathcal{M}} f \ : \ f \in H^s(\mathcal{M}_0)\},$$

where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$. For an open region ω_0, throughout the talk, $D(\omega_0)$ stands for the space of C^∞smooth test functions with compact support in ω_0. Denote $r_M \widetilde{H}(\mathcal{M}) =: \overset{o}{H}(\mathcal{M})$.

We formulate the following Problem

Problem (A): *Find vectors* $[\overset{0}{\widetilde{v}}], [\overset{0}{\widetilde{p}}]$ *in* $Z(\omega)$ *satisfying* *the differential equations* (10), (11);

the initial condition

$$\overset{0}{\widetilde{v}}(x,0)=0,\ x:=(x_1,x_2)\in\omega; \tag{12}$$

the boundary conditions on $Z(\ell)$

$$\{\overset{0}{\widetilde{v}}(x,t)\}^+=0\,,\quad (x,t)\in Z(\ell^{(D)}), \tag{13}$$

$$\{T(\partial,n)[\overset{0}{\widetilde{v}}(x,t),\,\overset{0}{\widetilde{p}}(x,t)]\}^+=g^{(N)}(x,t)\,,\quad (x,t)\in Z(\ell^{(N)}), \tag{14}$$

where the symbols $\{\,\cdot\,\}^+$ denote the interior one-sided limits with respect to the spatial variable x on ℓ.

Function spaces for the boundary and transmission data, and for solution vectors will be specified below.

For sufficiently smooth vector functions and smooth domains, by standard arguments we easily derive the following Green's formulas (see, e.g.,[5], [6], and also compare [4]):

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\omega}\rho\left|\overset{0}{\widetilde{v}}\right|^2dx+\int\limits_{\omega}2\mu\sum_{i,j=1}^{3}e^2_{ij0}(\overset{0}{\widetilde{v}})dx$$
$$-\int\limits_{\ell}\{T(\partial,n)[\overset{0}{\widetilde{v}},\overset{0}{\widetilde{p}}]\}^+\cdot\{\overset{0}{\widetilde{v}}\}^+\,d\ell=\int\limits_{\omega}[-\mu\,\Delta_2\overset{0}{\widetilde{v}}+\rho\,\dot{\overset{0}{\widetilde{v}}}+\operatorname{grad}\overset{0}{\widetilde{p}}]\cdot\overset{0}{\widetilde{v}}\,dx, \tag{15}$$

Remark that the above Green's formula by standard limiting procedure can be generalized to Lipschitz domains and to vector–functions from the corresponding Sobolev-Slobodetski and Bessel potential spaces. In particular, they remain valid if ℓ is Lipschitz surface and

$$\overset{0}{\widetilde{v}}(\cdot,t),\ \frac{\partial\overset{0}{\widetilde{v}}(\cdot,t)}{\partial x_\alpha},\ A[\overset{0}{\widetilde{v}},\overset{0}{\widetilde{p}}](\cdot,t)\in C^2([0,\infty),L_2(\omega)). \tag{16}$$

In this case, the surface integrals in Green's formula should be replaced by dualities $\langle\cdot,\cdot\rangle$ between the corresponding spaces. In addition to the conditions stated in the formulation of Problem (A), we require that solution vector $[\overset{0}{\widetilde{v}},\overset{0}{\widetilde{p}}]$ satisfy imbedding relations (16).

The following uniqueness result can be easily proved:

Theorem 2.1. *The homogeneous mixed Problem (A) has only the trivial solution in the space of vector functions satisfying the conditions (16).*

We recall that for an original function f satisfying the growth condition $|f(t)|\le c\,e^{at}$ with $a\in\mathbb{R}$, the Laplace transform reads as

$$\mathcal{L}[f]=\int\limits_0^\infty f(t)\,e^{-\tau t}\,dt,\quad \tau=\sigma_1+i\sigma_2\in\mathbb{C},\quad \Re\,\tau=\sigma_1>a,$$

and if $f\in C^k([0,+\infty))$, then

$$\mathcal{L}[f^{(k)}]=\int\limits_0^\infty f^{(k)}(t)\,e^{-\tau t}\,dt=\tau^k\mathcal{L}[f]-f(0)\tau^{k-1}-f'(0)\tau^{k-2}-\,\cdots\,-f^{(k-1)}(0). \tag{17}$$

Let us apply formally the Laplace transform to the initial-boundary-transmission dynamical problem (A) in order to reduce it to an elliptic problem depending on the complex parameter τ. To this end, we introduce the notation

$$V(x,\tau) := \mathcal{L}[\overset{0}{\widetilde{v}}(x,\,\cdot)], \quad P(x,\tau) := \mathcal{L}[\overset{0}{\widetilde{p}}(x,\,\cdot)]. \tag{18}$$

Taking into account the homogeneous initial conditions (12) and formula (17), we get from (10) and (11):

$$\mathcal{A}[V,P] := -\mu\Delta_2 V + \varrho\,\tau\,V + \operatorname{grad} P = X \text{ in } \omega, \tag{19}$$

$$\operatorname{div} V = 0 \text{ in } \omega, \tag{20}$$

where $V = (V_1, V_2, V_3)$, and

$$X(x,\tau) := \mathcal{L}[F(x,\,\cdot)]\,.$$

Analogously, boundary conditions (13), (14) can be written as

$$\{V\}^+ = 0 \text{ on } \ell^{(D)}, \tag{21}$$

$$\{T(\partial,n)[V,P]\}^+ = G^{(N)} \text{ on } \ell^{(N)}\,, \tag{22}$$

where the stress operator $T(\partial,n)$ is defined by (2) and

$$G^{(N)}(x,\tau) := \mathcal{L}[g^{(N)}(x,\cdot)].$$

Thus, by this approach we have reduced formally the original *dynamical initial boundary problem* (A) to the elliptic problem (19)-(22) depending on the complex parameter τ, which we refer as the *pseudo-oscillation boundary* **problem (B)**.

In what follows, first we investigate the problem (B), prove the uniqueness and existence results, analyze the regularity of solutions and its dependence on the parameter τ and afterwards we return to the original dynamical problem via the inverse Laplace transform.

Uniqueness

We look for solution vectors $[V,P]$ of problem (B) in the Bessel potential spaces, in particular, we assume that

$$V(\cdot\,,\tau) \in H^1(\omega), \quad P(\cdot\,,\tau) \in L_2(\omega)\,, \tag{23}$$

and

$$X(\cdot\,,\tau) \in L_2(\omega), \quad G^{(N)}(\cdot\,,\tau) \in H^{-1/2}(\ell^{(N)})\,. \tag{24}$$

In this case, the Dirichlet type boundary and transmission conditions are understood in the usual trace sense, while the Neumann type conditions are understood in the generalized functional sense. Let us recall that the generalized trace functionals for the stress vectors, corresponding to vector $V \in H^1(\omega)$ with

$$\mathcal{A}V \in L_2(\omega), \quad \operatorname{div} V = 0, \tag{25}$$

are defined with the help of the relations (Green's identities)

$$\begin{aligned} &\langle\{(T(\partial,n)[V,P])_j\}^+, \{\overline{\widetilde{V}_j}\}^+\rangle_\ell \\ &\quad := \int\limits_\omega [\,\varrho\,\tau\,V_j\,\overline{\widetilde{V}_j} + 2\mu\,e_{kj}(V)\,e_{kj}(\overline{\widetilde{V}})\,]\,dx - \int\limits_\omega [\mathcal{A}[V,P]]_j\,\overline{\widetilde{V}_j}\,dx. \end{aligned} \tag{26}$$

Here $\widetilde{V} \in H^1(\omega)$ with $\mathrm{div}\widetilde{V} = 0$ in Ω are arbitrary vectors. The symbol $\langle \cdot\,,\,\cdot \rangle_{\mathcal{M}}$ denotes the duality relation between the spaces $H^{-1/2}(\mathcal{M})$ and $H^{1/2}(\mathcal{M})$, which generalizes the usual $L_2(\mathcal{M})$ inner product. By these relations the generalized traces of the stress vector $\{T(\partial, n)[V, P]\}^+ \in H^{-1/2}(\ell)$ are correctly determined. Note that for arbitrary vector functions $X = (X_1, X_2, X_3) \in L_2(\mathcal{M})$ and $Y = (Y_1, Y_2, Y_3) \in L_2(\mathcal{M})$ we have

$$\langle\, X\,,\, Y \,\rangle_{\mathcal{M}} = \int_{\mathcal{M}} X_j\, Y_j\, d\mathcal{M}.$$

The investigation of the problem under consideration we start with the following uniqueness result.

Theorem 2.2. *Let $\tau = \sigma_1 + i\sigma_2$ with $\sigma_1 > \sigma_0 > 0$. Then the homogeneous problem (B) has only the trivial solution.*

Proof. From (26) we obtain

$$\int_{\omega} [\, \varrho\, \tau\, V_j\, \overline{\widetilde{V}_j} + 2\mu\, e_{kj}(V)\, e_{kj}(\,\overline{\widetilde{V}}\,)\,]\, dx$$
$$= \int_{\omega} [-\mu\Delta_2 V + \varrho\, \tau\, V + \mathrm{grad}\, P]_j\, \overline{\widetilde{V}_j}\, dx - \langle \{(T(\partial, n)[V, P])_j\}^+, \{\overline{\widetilde{V}_j}\}^+ \rangle_{\ell}. \tag{27}$$

Now, let $[V, P]$ be a solution of the homogeneous problem (B). Evidently, the conditions (23) and (25) are satisfied and we can apply the generalized Green formulas (15) and (26), and the resulting relation (27). From (27) with $\widetilde{V} = V$, in view of the homogeneous boundary-transmission conditions of the problem, we derive

$$\int_{\omega} [\, \varrho\, \tau\, V_j\, \overline{\widetilde{V}_j} + 2\mu\, e_{kj}(V)\, e_{kj}(\,\overline{\widetilde{V}}\,)\,]\, dx = 0. \tag{28}$$

Separate the real part of the equality (28)

$$\int_{\omega} [\, \varrho\, \sigma_1\, V_j\, \overline{\widetilde{V}_j} + 2\mu\, e_{kj}(V)\, e_{kj}(\,\overline{\widetilde{V}}\,)\,]\, dx = 0. \tag{29}$$

Taking into account the inequalities $\sigma_1 > 0$, from (29) we conclude that $V = 0$ in ω. Then it follows from (19) that $P = C = \mathit{const}$ in ω and, therefore, $\{T(\partial, n)[V, P]\} = -C\, n$ on ℓ. Finally, the homogeneous Neumann condition (22) with $G^{(N)} = 0$ on $\ell^{(N)}$ implies $C = 0$, which completes the proof. □

Existence

Recall that $\bar{\omega} = \omega \cup \ell$ and define the vector-function space

$$\mathbb{H}^1(\omega, \ell^{(D)}) := \Big\{ \Phi = (\Phi_1, \Phi_2, \Phi_3) \in H^1(\omega) \,:\, \{\Phi\}^+_{\ell^{(D)}} = 0,\, r_{_\omega} \Phi = \Phi,\, \mathrm{div}\, \Phi = 0 \Big\}. \tag{30}$$

We recall also that $r_{_\mathcal{M}}$ stands for the restriction operator on $\mathcal{M}$. The inner product and the norm in the space $\mathbb{H}^1(\omega, \ell^{(D)})$ by definition coincide with the inner product and the norm of the space $H^1(\omega)$. Therefore, $\mathbb{H}^1(\omega, \ell^{(D)})$ is a Hilbert space.

Further, let W, $\widetilde{W} \in \mathbb{H}^1(\omega, \ell^{(D)})$ and denote, for convenience and to adopt the notation to our basic problem (B),

$$W = r_{_\omega} W \in H^1(\omega), \;\; \widetilde{W} = r_{_\omega} \widetilde{W} \in H^1(\omega). \tag{31}$$

and introduce the sesquilinear form

$$\begin{aligned} \mathcal{B}_\tau(W, \widetilde{W}) \;\; &:= \int_\omega [\, \varrho\, \tau\, W_j \, \overline{\widetilde{W}_j} + 2\mu\, e_{kj}(W)\, e_{kj}(\, \overline{\widetilde{W}}\,)\,]\, dx \\ &= \int_\omega [\, \varrho\, \tau\, V_j \, \overline{\widetilde{V}_j} + 2\mu\, e_{kj}(V)\, e_{kj}(\, \overline{\widetilde{V}}\,)\,]\, dx. \end{aligned} \tag{32}$$

Next we introduce the anti-linear form

$$\begin{aligned} \mathcal{F}_\tau(\widetilde{W}) \;\; &:= \int_\omega X_j \, \overline{\widetilde{W}_j}\, dx + \langle\, G_j^{(N)},\, \{\overline{\widetilde{W}_j}\}^+ \,\rangle_{\ell^{(N)}} \\ &= \int_\omega X_j \, \overline{\widetilde{V}_j}\, dx + \langle\, G_j^{(N)},\, \{\overline{\widetilde{V}_j}\}^+ \,\rangle_{\ell^{(N)}} \,. \end{aligned} \tag{33}$$

where $\langle \cdot, \cdot \rangle_{\ell^{(N)}}$ denotes the duality between the spaces $H^{-1/2}(\ell^{(N)})$ and $\widetilde{H}^{1/2}(\ell^{(N)})$. The duality is well defined since $G^{(N)} \in H^{-1/2}(\ell^{(N)})$ and $\{\widetilde{W}\}^+_\ell = \{\widetilde{V}\}^+_\ell \in \widetilde{H}^{1/2}(\ell^{(N)})$. By the Schwartz inequality and the trace theorem one can easily show that the functional $\mathcal{F}_\tau : \mathbb{H}^1(\omega, \ell^{(D)}) \to \mathbb{C}$ is continuous.

Theorem 2.3. *Let $\tau = \sigma_1 + i\sigma_2$ with $\sigma_1 > \sigma_0 > 0$. The sesquilinear form*

$$\mathcal{B}_\tau : \mathbb{H}^1(\omega, \ell^{(D)}) \times \mathbb{H}^1(\omega, \ell^{(D)}) \to \mathbb{C}$$

is bounded and coercive, i.e., there are positive constants $C_1(\tau)$ and $C_2(\tau)$ such that

$$|\, \mathcal{B}_\tau(W, \widetilde{W})\, | \le C_1(\tau)\, ||\, W\, ||_{H^1(\omega)}\, ||\, \overline{\widetilde{W}}\, ||_{H^1(\omega)}, \tag{34}$$

$$\Re\, [\, \mathcal{B}_\tau(W, W)\,] \ge C_2(\tau)\, ||\, W\, ||^2_{H^1(\omega)}, \tag{35}$$

for all $W, \widetilde{W} \in \mathbb{H}^1(\omega, \ell^{(D)})$.

Proof. By Schwartz inequality from (32) we derive

$$\begin{aligned} |\, \mathcal{B}_\tau(W, \widetilde{W})\, | \;\; &\le \;\; c_3\, |\tau|\; ||\, V\, ||_{L_2(\omega)}\, ||\, \widetilde{V}\, ||_{L_2(\omega)} + c_4\, ||\, V\, ||_{H^1(\omega)}\, ||\, \widetilde{V}\, ||_{H^1(\omega)} \\ &\le \;\; c_4\, ||\, W\, ||_{H^1(\omega)}\, ||\, \widetilde{W}\, ||_{H^1(\omega)} + c_3\, |\tau|\; ||\, W\, ||_{L_2(\omega)}\, ||\, \widetilde{W}\, ||_{L_2(\omega)} \\ &\le \;\; \Big[c_3\, |\, \tau\, | \, + c_4 \Big] ||\, W\, ||_{H^1(\omega)}\, ||\, \widetilde{W}\, ||_{H^1(\omega)}, \end{aligned} \tag{36}$$

which proves (34) with

$$C_1(\tau) = c_3\, |\, \tau\, | \, + c_4. \tag{37}$$

Here the positive constants c_k, $k = \overline{1, 4}$, do not depend on τ.

Now we show the coercivity property.

Taking the real pert of (32) and applying (36), we get

$$
\begin{aligned}
\Re\,[\,\mathcal{B}_\tau(W,W)\,] &= \int\limits_\omega [\,2\mu\, e_{kj}(V)\, e_{kj}(\overline{V}) + \varrho\,\sigma_1\, V_j\, \overline{V_j}\,]\, dx \\
&= \int\limits_\omega \Big[2\mu \sum_{k,j=1}^{3} |e_{kj}(W)|^2 + \varrho\,\sigma_1\, |W|^2 \Big]\, dx \\
&\geq \int\limits_\omega \Big[2\mu \sum_{k,j=1}^{3} |e_{kj}(W)|^2 + \varrho\,\sigma_1\, |W|^2 \Big]\, dx \\
&\geq \int\limits_\omega \Big[c_5(\tau) \sum_{k,j=1}^{3} |e_{kj}(W)|^2 + c_6(\tau)\, |W|^2 \Big]\, dx,
\end{aligned}
$$

where

$$c_5(\tau) = 2\,\mu > 0, \quad c_6(\tau) = \sigma_1\,\varrho > 0. \tag{38}$$

Since $W \in H^1(\omega)$ and $\{W\}^+_{\ell(D)} = 0$ by the well known Korn's inequality we have (see, e.g., [7])

$$\int\limits_\omega \sum_{k,j=1}^{3} |e_{kj}(W)|^2\, dx \geq c_7\, ||W||^2_{H^1(\omega)}, \tag{39}$$

where c_7 is a positive constant depending only on the geometrical parameters of the domain ω. Finally we arrive at the inequality

$$\Re\,[\,\mathcal{B}_\tau(W,W)\,] \geq c_5(\tau)\, c_7\, ||W||^2_{H^1(\omega)} + c_6(\tau)\, ||W||^2_{L_2(\omega)}, \tag{40}$$

which proves (35) with

$$C_2(\tau) = c_7\, 2\,\mu, \tag{41}$$

where c_7 does not depend on τ. This completes the proof. □

Now let us consider the following variational problem: *Find a vector* $W \in \mathbb{H}^1(\omega, \ell^{(D)})$ *satisfying the equation*

$$\mathcal{B}_\tau(W, \widetilde{W}) = \mathcal{F}_\tau(\widetilde{W}) \text{ for all } \widetilde{W} \in \mathbb{H}^1(\omega, \ell^{(D)}). \tag{42}$$

We have the following existence results.

Theorem 2.4. *Let* $\tau = \sigma_1 + i\sigma_2$ *with* $\sigma_1 > \sigma_0 > 0$. *The variational problem* (42) *is uniquely solvable and for the solution vector there holds the inequality*

$$||W||^2_{H^1(\omega)} \leq C_3(\tau) \Big(||X||_{L_2(\omega)} + ||G^{(N)}||_{H^{-1/2}(\ell^{(N)})} \Big), \tag{43}$$

where $C_3(\tau)$ *is a positive constant depending on* τ *and on the material parameters, and*

$$0 < C_3(\tau) \leq \frac{|\tau|^2}{\sigma_1}\, C_4 \leq \frac{|\tau|^2}{\sigma_0}\, C_4 \tag{44}$$

with C_4 *independent of* τ.

Proof. Existence and uniqueness are the direct consequences of the Theorem 1.3 and the Lax-Milgram theorem since the functional $\mathcal{F}_\tau$ given by (33) is bounded. Indeed,

$$\begin{aligned} |\,\mathcal{F}_\tau(W)\,| &\le \;||\,X\,||_{L_2(\omega)}\,||\,V\,||_{L_2(\omega)} + ||G^{(N)}||_{H^{-1/2}(\ell^{(N)})}\,||r_{\ell^{(N)}}\,\{V\}^+||_{H^{1/2}(\ell^{(N)})} \\ &\le \;(\,||\,X\,||_{L_2(\omega)}\,)\,||\,W\,||_{L_2(\omega)} + ||G^{(N)}||_{H^{-1/2}(\ell^{(N)})}\,||\{W\}^+||_{H^{1/2}(\ell)} \\ &\le \;(\,||\,X\,||_{L_2(\omega)}\,)\,||\,W\,||_{H^1(\omega)} + \delta_1\,||G^{(N)}||_{H^{-1/2}(\ell^{(N)})}\,||\,W\,||_{H^1(\omega)} \\ &\le \;(\,||\,X\,||_{L_2(\omega)} + \delta_1\,||G^{(N)}||_{H^{-1/2}(\ell^{(N)})}\,)\,||\,W\,||_{H^1(\omega)}, \end{aligned}$$

where δ_1 depends only on the geometry of the domain ω and corresponds to the trace estimate, i.e.,

$$||\{W\}^+||_{H^{1/2}(\ell)} \le \delta_1\,||\,W\,||_{H^1(\omega)}. \tag{45}$$

Now, the inequality (35) completes the proof with $C_3(\tau) = [\,C_2(\tau)\,]^{-1}\delta_2$, where $\delta_2 = \max\{1, \delta_1\}$. □

Further, we prove the following assertion.

Theorem 2.5. *Let $W \in \mathbb{H}^1(\omega, \ell^{(D)})$ solve the variational problem (42) and let*

$$V := r_{_\omega}\,W. \tag{46}$$

Then there exists a unique function $P \in L_2(\omega)$, such that the vector $[V, P]$ solve the problem (B).

Proof. Let us assume that V^f solve the variational problem, i.e.,

$$\int\limits_\omega [\,\varrho\,\tau\,V_j\,\overline{\widetilde{V}_j} + 2\mu\,e_{kj}(V)\,e_{kj}(\,\overline{\widetilde{V}}\,)\,]\,dx = \int\limits_\omega X_j\,\overline{\widetilde{V}_j}\,dx + \langle\,G_j^{(N)},\,\{\overline{\widetilde{V}_j}\}^+\,\rangle_{\ell^{(N)}}\,. \tag{47}$$

We have to verify the relations (19)-(22). Note that the conditions (21) and (22) are satisfied automatically due to the definition of the space $\mathbb{H}^1(\omega, \ell^{(D)})$.

We start with the differential equations (19) and (20). The arguments are standard. Taking in (42) that

$$\widetilde{W} = \widetilde{V} \in \mathcal{D}(\omega) \cap \mathbb{H}^1(\omega, \ell^{(D)})$$

we arrive at the equations:

$$\mathcal{B}_\tau(W, \widetilde{V}) \;\equiv\; \int\limits_\omega [\,\varrho\,\tau\,W_j\,\overline{\widetilde{V}_j} + 2\mu\,e_{kj}(W)\,e_{kj}(\,\overline{\widetilde{V}}\,)\,]\,dx = \int\limits_\omega X_j\,\overline{\widetilde{V}_j}\,dx,$$

which, in view of (46), can be rewritten as

$$\int\limits_\omega [\,\varrho\,\tau\,V_j\,\overline{\widetilde{V}_j} + 2\mu\,e_{kj}(V)\,e_{kj}(\,\overline{\widetilde{V}}\,)\,]\,dx = \int\limits_\omega X_j\,\overline{\widetilde{V}_j}\,dx.$$

In turn these equations are equivalent to the following distributional relations

$$\langle -\mu\,\Delta V + \varrho\,\tau\,V - X\,,\,\widetilde{V}\rangle_\omega = 0. \tag{48}$$

The from (48) it follows that there exists a function $P_1 \in L_2(\omega^f)$ such that (19) holds in the distributional sense (for details see [8], Ch. I, Subsection).

The equation (19) is understood in the distributional sense. Note that the function P_1 is defined modulo a constant summand.

To show the condition (21) we proceed as follows.

From Green's formulas (15) and (26) we easily derive

$$\langle\{(T(\partial,n)[V,P_1])_j\},\{\overline{\widetilde{V_j}}\}\rangle_\ell = \int\limits_\omega [\,\varrho\,\tau\, V_j\, \overline{\widetilde{V_j}} + 2\mu\, e_{kj}(V)\, e_{kj}(\overline{\widetilde{V}})\,]\, dx - \int\limits_\omega X_j\, \overline{\widetilde{V_j}}\, dx. \tag{49}$$

Since $\operatorname{div}\widetilde{V} = 0$ in ω, we have

$$\int\limits_{\ell^{(N)}} \{\widetilde{V}\}^+ \cdot n\, d\ell = 0, \tag{50}$$

where n is the outward normal to ℓ. Therefore, from equation (50) we conclude

$$\{(T(\partial,n)[V,P_1])_j\}^+ = G_j^{(N)} + C_1\, n_j, \ \ j = 1,2,3, \ \text{ on } \ \ell^{(N)}, \tag{51}$$

where C_1 is an arbitrary constant.

Hence,

$$\{(T(\partial,n)[V,P_1 + C])\}^+ = G^{(N)} \ \text{ on } \ \ell^{(N)}, \tag{52}$$

Thus, we have shown that if V solve the variational problem (42), then $[V,P]$ with $P = P_1 + C$ solve the problem (B). Due to the uniqueness Theorem 1.2, we conclude that the pressure function $P = P_1 + C$ is defined uniquely. □

Existence results for the dynamical problem.

Here we apply the inverse Laplace transform and construct a solution to the original dynamical problem. Let $V(x,\tau)$, and $P(x,\tau)$ be a solution to the Problem (B) whose existence and uniqueness we have shown in the previous subsection.

To demonstrate our approach, for simplicity we assume that the data $F(x,t)$, and $g^{(N)}(x,t)$ of the original dynamical Problem (A) are C^∞-smooth in the regions: $\overline{\omega} \times [0,+\infty)$, and $\overline{\ell^{(N)}} \times [0,+\infty)$, respectively, vanish identically for $t \in [0,\varepsilon)$ with some positive ε and are polynomially bounded in t as $t \to \infty$. Then it follows that their Laplace transforms $X(x,\tau)$ and $G^{(N)}(x,\tau)$, which are the data of the BVP (B), are analytic with respect to τ in the complex half plane $\Re\tau = \sigma_1 > 0$ and decay at infinity faster than any power of $|\tau|^{-1}$.

$V(\cdot,\tau)$ and $P(\cdot,\tau)$ are analytic with respect to τ in the same complex half plane $\Re\tau = \sigma_1 > 0$ and by one of the above

$$||V(\cdot,\tau)||_{H^1(\omega)} \le C\,|\tau|^{-m}, \tag{53}$$

where the positive constant C do not depend on τ and m is an arbitrary natural number.

Since the pair $[V(\cdot,\tau), P(\cdot,\tau)]$ solves equations (19)-(20), we have

$$\Delta\, P(x,\tau) = \operatorname{div} X(x,\tau), \quad\ x \in \omega. \tag{54}$$

Therefore, if $\operatorname{div} X(\cdot,\tau) \in H^r(\omega)$, then $P(\cdot,\tau) \in H^{r+2}(\omega_*)$ for arbitrary $r \geq 0$ and an arbitrary proper subdomain ω_* of ω, such that $\overline{\omega_*} \subset \omega$. In particular, if $\operatorname{div} X(\cdot,\tau) \in C^\infty(\overline{\omega})$, then $P(\cdot,\tau) \in C^\infty(\omega) \cap L_2(\omega)$.

Moreover, in addition, if we assume that the surfaces ℓ and $\ell := \partial\ell^{(D)}$ are C^∞-smooth, then by the interior and boundary regularity results, we have the imbedding ([8]):

$$\begin{aligned} V(\cdot,\tau) &\in C^\infty(\overline{\omega} \setminus \ell) \cap H^1(\omega), \\ P(\cdot,\tau) &\in C^\infty(\overline{\omega} \setminus \ell) \cap L_2(\omega). \end{aligned} \tag{55}$$

One can easily show that

$$||P(\cdot,\tau)||_{L_2(\omega)} \leq C_* \, |\tau|^{-m}, \qquad ||P(\cdot,\tau)||_{H^1(\widetilde{\omega}_*)} \leq C_* \, |\tau|^{-m}, \tag{56}$$

where the positive constant C_* does not depend on τ, m is an arbitrary natural number and $\overline{\widetilde{\omega}_*} \subset [\,\overline{\omega} \setminus \ell\,]$.

Denote

$$v(x,t) = \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} V(x,\tau)\, e^{\tau t}\, d\tau, \quad x \in \omega, \tag{57}$$

$$p(x,t) = \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} P(x,\tau)\, e^{\tau t}\, d\tau, \quad x \in \omega. \tag{58}$$

By the Minkowski inequality

$$\Big[\int_{\omega_1} \Big\{ \int_{\omega_2} |f(x,y)|\, dy \Big\}^p dx \Big]^{\frac{1}{p}} \leq \int_{\omega_2} \Big\{ \int_{\omega_1} |f(x,y)|^p\, dx \Big\}^{\frac{1}{p}} dy, \quad p > 1,$$

we get

$$||v(\cdot,t)||_{H^1(\omega)} \leq \frac{e^{\sigma_1 t}}{2\pi} \int\limits_{-\infty}^{\infty} ||V(\cdot,\tau)||_{H^1(\omega)}\, d\sigma_2 < \infty, \tag{59}$$

$$||p(\cdot,t)||_{L_2(\omega)} \leq \frac{e^{\sigma_1 t}}{2\pi} \int\limits_{-\infty}^{\infty} ||P(\cdot,\tau)||_{L_2(\omega)}\, d\sigma_2 < \infty. \tag{60}$$

Now, let us check that the pair $[v(x,t), p(x,t)]$, defined by (57)-(58), solve the original problem (A).

Indeed, the differential equations (10)-(11) can be verified by direct differentiation. The boundary conditions (13)-(14) can be obtained by taking the corresponding traces on the interface and on the boundary.

It remains to show the initial conditions hold. By passing to the limit as $t \to 0$, from (57) we have

$$v(x,0) = \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} V(x,\tau)\, d\tau. \tag{61}$$

Now, by the decay condition (53), we derive that $v(\cdot,0) = 0$ in ω.

Remark 2.6. For Lipschitz boundaries, in particular, for piecewise smooth boundaries the same formulas (57)-(58) give solutions to the dynamical problem in the appropriate sense.

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