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Quasistatic Equations of Various Orders of Approximation of the Second Strain Gradient Theory of Elastic Prismatic Bodies

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A vector equation of motion of the second strain and velocity vector gradient linear elasticity theory with respect to the displacement vector for an arbitrary anisotropic body is given. From this motion equation the motion equation with respect to the displacement vector is obtained in the case of an isotropic medium, from which, in turn, the decomposed motion equation with respect to the displacement vector is given. Further, from the last decomposed motion equation the quasi-static equation of the second strain gradient of the linear theory of thin prismatic bodies with respect to the displacement vector is derived under the classical parameterization of the body domain. From the last quasi-static equation, the infinite system of equations in the moments of the displacement vector with respect to an arbitrary system of orthogonal polynomials is obtained. From the infinite system of equations in the moments of the displacement vector with respect to the system of Legendre orthogonal polynomials, the equations of several first approximations (from the zeroth to the twentieth approximation) are derived. In this paper, the system of equations in the moments of the tenth approximation is considered in more detail. In particular, like any system with any order of approximation, it splits into two systems, one of which contains the moments of the displacement vector of an even order, and the other contains the moments of an odd order. Each of these systems, in turn, is decomposed and for each moment included in these systems of equations, a separate equation of elliptic type of higher order is obtained, for which a representation of the analytical solution can be written using the Vekua method.

It is noteworthy that the methodologies outlined above can be readily extended to accommodate other rheological bodies, including higher-order gradient theories, as well as both linear and non-linear classical and micropolar theories of elastic and viscoelastic bodies.

Keywords and phrases: Thin body, deformation tensor, stress tensor, gradient theory, the second strain tensor and velocity vector gradient theory, method of orthogonal polynomials.

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Introduction

Strain gradient elasticity theories and similar nonlocal theories have appeared in the literature (see [1]), as analytical tools capable of taking into account long-range interaction forces in the microstructure of materials and recording experimentally discovered phenomena (such as length scale effects, dispersion waves, localization of deformation, etc.), which classical theory is not able to take into account (see, for example, [2, 3, 4, 5, 6]). Strain gradient theories do not take into account the independent rotations of material particles, i.e., they are not polar in nature,

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and the moment stress tensor arises due to the gradient of the vortex vector of the continuous medium. In this context, of particular interest is the most important article by Mindlin [5], who constructed a gradient theory of second-order elasticity with respect to the strain tensor for statics, as well as for dynamics [4, 6]. Although it uses an excessive number of material constants, it is a benchmark because of its conceptual foundations and ability to capture length scale effects. R. Toupin and D.C. Gazis [7], exploring models of mono- and di-atomic lattices, pointed out the ability of the second strain gradient theory to interpret the atomic structure of matter.

Subsequently, Aifantis and his co-authors proposed a first-order gradient theory of elasticity, which used only three material constants (two classical Lame constants and a length scale parameter) [8, 9, 10]. This theory was then extended to dynamics of higher order inertia [11, 12], which requires only one additional length scale parameter. This model of gradient elasticity, called the Aifantis model, has been used to solve a variety of structural problems, showing that strain singularities do not arise at crack tips and dislocation cores, and that dispersion effects in wave propagation can be captured [12]. However, as shown by Lazar and Maugin [13], the singularities of the double stress tensor cannot be eliminated using the mentioned Aifantis model.

An extension of the above first strain gradient theory to a second strain gradient one, characterized by four material constants – two Lame constants and two length scale parameters, was proposed in [14]. A similar theory, called bi-Helmholtz type strain gradient theory, was independently developed by Lazar and Maugin [15, 16] and applied to a number of dislocation problems in an infinite domain with the outstanding result that no singularities arise in the dislocation core and, of course, this material model has a high regularization ability. This result was largely confirmed by a number of applications to problems of defect interaction [17], dislocation analysis [15, 16, 18] and disclination analysis [19]. However, no attention has been paid to the boundary conditions and surface effects that typically characterize recent gradient theory models. The role of velocity gradients and their inherent higher order inertia terms in the equations of motion has been considered in a series of studies dealing with wave motion and related dispersion phenomena. This topic has been widely covered in the literature, but here we will limit ourselves to mentioning the following works: [11, 20, 21, 22, 23, 24, 25, 26], see also review article [12] and references therein. Recent literature suggests that higher order inertia models are capable of realistically describing wave dispersion phenomena. In the works [11, 24, 25] the authors put forward the concept of a "dynamically consistent" gradient model, namely a model endowed with gradient improvements in both stiffness and inertia characteristics, which makes it possible to eliminate deformation field singularities arising from e.g. near the sharp tip of a crack and realistically describe the dispersion characteristics of wave propagation in an inhomogeneous medium. However, the higher order inertial terms appearing in the governing equations were introduced heuristically in these studies, and their relationship to kinetic energy remained unclear.

In the works [27, 28] the authors provided useful information about the laws conservation and balance for gradient elastodynamics. A review of the historical developments of gradient elasticity and higher order inertia with relevant applications can be seen in [12]. It should be noted that the above brief overview of gradient theories is borrowed mainly from the work [1] and it is also given in [29]. It should be especially noted that the works of S.A. Lurie and his co-authors deserve great attention [30, 31, 32]. Note that by extending the work [14] and, of course, the works mentioned above are the works [1, 33], which set out some of the most important issues of the second strain tensor and velocity vector gradient elasticity theory. In particular, formulations of boundary-value and initial-boundary-value problems are given. We also note that some more general questions on the second strain and velocity vector gradient three-dimensional linear theory of elasticity and theory of thin bodies are discussed in [29]. In the work under consideration, some issues of the second strain gradient theory of prismatic thin elastic bodies are considered in more detail.

1 Motion equation of the second strain and velocity gradient linear elasticity theory for the displacement vector for an arbitrary anisotropic body

The vector motion equation of the second strain and velocity vector gradient linear elasticity theory with respect to the displacement vector for an arbitrary anisotropic body has the form [29, 34]

$$\underset{\approx}{\mathbf{M}}' \cdot \mathbf{u} + \rho \mathbf{F} = 0, \tag{1}$$

where the following designations are introduced

$$\begin{aligned} &\underset{\approx}{\mathbf{M}}' = \underset{\approx}{\mathbf{L}}' - \rho^* \underset{\approx}{\mathbf{E}} \partial_t^2, \quad \underset{\approx}{\mathbf{L}}' = a \underset{\approx}{\mathbf{L}}, \\ &a = 1 - \underset{\approx}{\mathbf{a}}^{(1)} \overset{2}{\otimes} \nabla\nabla + \underset{\approx}{\underset{\approx}{\mathbf{a}}}^{(2)} \overset{4}{\otimes} \nabla\nabla\nabla\nabla, \ \rho^* = \rho b, \\ &b = 1 - \underset{\approx}{\mathbf{b}}^{(1)} \overset{2}{\otimes} \nabla\nabla + \underset{\approx}{\underset{\approx}{\mathbf{b}}}^{(2)} \overset{4}{\otimes} \nabla\nabla\nabla\nabla. \end{aligned} \tag{2}$$

Note that (1) represents the equations of motion of the second-order gradient theory with respect to the deformation tensor and the velocity vector for any anisotropic bodies. Here $\underset{\approx}{\mathbf{M}}'$ is the tensor-operator of the second rank and the sixth order of the motion equations (1), $\underset{\approx}{\mathbf{L}} = \mathbf{e}_i \mathbf{e}_l A_{ijkl} \partial_j \partial_k$ is the tensor-operator of the second rank and of the second order of the classical quasi-static equation of any anisotropic bodies, $\nabla = \mathbf{e}_i \partial_i$ is a Hamilton vector-operator, $\underset{\approx}{\underset{\approx}{\mathbf{A}}} = A_{ijkl} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l$ is an elastic modulus tensor, $A_{ijkl} = A_{klij} = A_{jikl}$, $\underset{\approx}{\mathbf{a}}^{(1)} = a_{ij}^{(1)} \mathbf{e}_i \mathbf{e}_j$ and $\underset{\approx}{\underset{\approx}{\mathbf{a}}}^{(2)} = a_{ijkl}^{(2)} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l$ are length scale tensors for statics, $a_{ij}^{(1)} = a_{ji}^{(1)}$, $a_{ijkl}^{(2)} = a_{klij}^{(2)} = a_{jikl}^{(2)}$, $\underset{\approx}{\mathbf{b}}^{(1)} = b_{ij}^{(1)} \mathbf{e}_i \mathbf{e}_j$ and $\underset{\approx}{\underset{\approx}{\mathbf{b}}}^{(2)} = b_{ijkl}^{(2)} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l$ are tensors for inertia effects, $b_{ij}^{(1)} = b_{ji}^{(1)}$, $b_{ijkl}^{(2)} = b_{klij}^{(2)} = b_{jikl}^{(2)}$, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, δ_{ij} is Kronecker delta, $i, j, k, l = 1, 2, 3$, ρ is the density of the material, $a = 1 - \underset{\approx}{\mathbf{a}}^{(1)} \overset{2}{\otimes} \nabla\nabla + \underset{\approx}{\underset{\approx}{\mathbf{a}}}^{(2)} \overset{4}{\otimes} \nabla\nabla\nabla\nabla$ is a scalar differential operator of the fourth order with length scale tensors, $b = 1 - \underset{\approx}{\mathbf{b}}^{(1)} \overset{2}{\otimes} \nabla\nabla + \underset{\approx}{\underset{\approx}{\mathbf{b}}}^{(2)} \overset{4}{\otimes} \nabla\nabla\nabla\nabla$ is a scalar differential operator of the fourth order with tensors for inertia effects, $\overset{r}{\otimes}$ is the inner r-product (see in [36, 35]). On repeating indices summation occurs from one to three.

1.1 Motion equation of the second strain and velocity gradient linear theory for isotropic homogeneous elastic media

In this case, the tensors considered above are represented as

$$\begin{aligned} &\underset{\approx}{\underset{\approx}{\mathbf{A}}} = a_1 \underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(1)} + a_2 (\underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(2)} + \underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(3)}), \ \underset{\approx}{\mathbf{a}}^{(1)} = a^{(1)} \underset{\approx}{\mathbf{E}}, \ \underset{\approx}{\mathbf{b}}^{(1)} = b^{(1)} \underset{\approx}{\mathbf{E}}, \\ &\underset{\approx}{\underset{\approx}{\mathbf{a}}}^{(2)} = a_1^{(2)} \underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(1)} + a_2^{(2)} (\underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(2)} + \underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(3)}), \ \underset{\approx}{\underset{\approx}{\mathbf{b}}}^{(2)} = b_1^{(2)} \underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(1)} + b_2^{(2)} (\underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(2)} + \underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(3)}), \end{aligned} \tag{3}$$

where $\underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(1)}$, $\underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(2)}$ and $\underset{\approx}{\underset{\approx}{\mathbf{C}}}_{(3)}$ are isotropic tensors of the fours rank (see in [36, 35, 29])

Taking into account (3), the differential operators included in (1) can be written in the form

$$\begin{aligned}
&\underset{\sim}{\mathbf{M}}' = \underset{\sim}{\mathbf{E}}Q_2' + (\lambda' + \mu')\nabla\nabla, \quad \underset{\sim}{\mathbf{M}}'_* = (\underset{\sim}{\mathbf{E}}Q_1' - (\lambda' + \mu')\nabla\nabla)Q_2' = \underset{\sim}{\mathbf{N}}'Q_2',\\
&\underset{\sim}{\mathbf{N}}' = \underset{\sim}{\mathbf{E}}Q_1' - (\lambda' + \mu')\nabla\nabla, \quad Q_1' = Q_2' + (\lambda' + \mu')\Delta = (\lambda' + 2\mu')\Delta - \rho^*\partial_t^2,\\
&Q_2' = \mu'\Delta - \rho^*\partial_t^2, \quad a = 1 - l_1^2\Delta + l_2^4\Delta^2, \quad \rho^* = \rho(1 - d_1^2\Delta + d_2^4\Delta^2),\\
&\underset{\sim}{\mathbf{N}}' \cdot \underset{\sim}{\mathbf{M}}' = \underset{\sim}{\mathbf{E}}Q_1'Q_2';\\
&Q_1' = [(\lambda + 2\mu)\Delta - \rho\partial_t^2] - \Delta[l_1^2(\lambda + 2\mu)\Delta - d_1^2\rho\partial_t^2]\\
&\qquad +\Delta^2(l_2^4(\lambda + 2\mu)\Delta - d_2^4\rho\partial_t^2],\\
&Q_2' = (\mu\Delta - \rho\partial_t^2) - \Delta(l_1^2\mu\Delta - d_1^2\rho\partial_t^2) + \Delta^2(l_2^4\mu\Delta - d_2^4\rho\partial_t^2),\\
&\mu' = \mu a,\ \lambda' = \lambda a,\ l_1^2 = a^{(1)},\ l_2^4 = a_1^{(2)} + 2a_2^{(2)},\ d_1^2 = b^{(1)},\ d_2^4 = b_1^{(2)} + 2b_2^{(2)}.
\end{aligned} \tag{4}$$

Here λ and μ are the Lame parameters.

Replacing $\underset{\sim}{\mathbf{M}}'$ in (1) by its expression (see the first expression in (4)) and taking into account the last relation on the third line in (4), we obtain the desired vector equation in the form

$$\underset{\sim}{\mathbf{M}}' \cdot \mathbf{u} + \rho\mathbf{F} = 0, \quad \text{where} \quad \underset{\sim}{\mathbf{M}}' = \underset{\sim}{\mathbf{E}}Q_2' + (\lambda' + \mu')\nabla\nabla. \tag{5}$$

Now, applying the operator $\underset{\sim}{\mathbf{N}}'$ with single multiplication to the equation (see (5)) on the left, we will obtain the decomposed equation

$$Q_1'Q_2'\mathbf{u} + \underset{\sim}{\mathbf{N}}' \cdot (\rho\mathbf{F}) = \mathbf{0}. \tag{6}$$

Note that more detailed questions about the splitting of equations and boundary conditions can be found, for example, in [35, 37, 38, and others].

1.2 Quasi-static equation with respect to displacement vector of the second strain gradient linear theory for isotropic homogeneous elastic media

In the case of statics or quasi-statics, from (6), we have the following vector equation

$$a\Delta^2\mathbf{u} + \mathbf{G} = 0, \tag{7}$$

where

$$a = 1 - l_1^2\Delta + l_2^4\Delta^2, \quad \mathbf{G} = \frac{1}{(\lambda + \mu)\mu}\underset{\sim}{\mathbf{N}} \cdot (\rho\mathbf{F}), \quad \underset{\sim}{\mathbf{N}} = (\lambda + 2\mu)\Delta - (\lambda + \mu)\nabla\nabla.$$

The equation (7) shall be written in the expanded form as follows

$$(l_2^4\Delta^4 - l_1^2\Delta^3 + \Delta^2)\mathbf{u} + \mathbf{G} = 0. \tag{8}$$

1.2.1 Quasi-static equation with respect to displacement vector of the second strain gradient linear theory for isotropic prismatic homogeneous elastic media

Consider a prismatic body of constant thickness of $2h$. Let us take the middle surface as the base surface. Then the nabla and the Laplacian operators are represented as follows [35]:

$$\nabla\mathbb{F} = (\mathbf{r}^P\partial_P + \mathbf{r}^3\partial_3)\mathbb{F} = (\mathbf{r}^P\partial_P + h^{-2}\mathbf{n}\partial_3)\mathbb{F}, \quad -1 \le x^3 \le 1,$$
$$\triangle\mathbb{F}(g^{PQ}\partial_P\partial_Q + g^{33}\partial_3^2)\mathbb{F} = (\bar{\triangle} + h^{-2}\partial_3^2)\mathbb{F}, \quad \bar{\triangle} = g^{PQ}\partial_P\partial_Q, \tag{9}$$

where $\mathbf{r}^p$, $p = 1, 2, 3$, is the contravariant basis, the capital Latin indices take the values 1, 2, and the summation from one to two occurs over the repeating capital Latin indices.

Let us use the Laplacian representation (see third relation from (9) and write the equations (8) of prismatic bodies in the form

$$[(l_2\bar{\Delta}^2 - l_1\bar{\Delta} + 1)\bar{\Delta}^2 + h^{-2}(4l_2\bar{\Delta}^2 - 3l_1\bar{\Delta} + 2)\bar{\Delta}\partial_3^2$$
$$+h^{-4}(6l_2\bar{\Delta}^2 - 3l_1\bar{\Delta} - 1)\partial_3^4 + h^{-6}(4l_2\bar{\Delta} - l_1)\partial_3^6 + 2h^{-8}l_2\partial_3^8]\mathbf{u} + \mathbf{G} = 0. \tag{10}$$

1.2.2 Quasi-static equations of the second strain gradient linear theory for isotropic prismatic homogeneous elastic media in moments of displacement vector

Applying the kth order moment operator of any orthogonal polynomial system (Legendre, Chebyshev) to equations (10), we obtain the following equation in the moments of the displacement vector:

$$(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(k)}{\mathbf{u}} + h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(k)}{\mathbf{u}}'' + h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)\overset{(k)}{\mathbf{u}}^{IV}$$
$$+h^{-6}(4l_2^4\bar{\Delta} - l_1^2)\overset{(k)}{\mathbf{u}}^{VI} + 2h^{-8}l_2^4\overset{(k)}{\mathbf{u}}^{VIII} + \mathbf{G} = 0, \quad k\in\mathbb{N}_0 \ (\mathbb{N}_0=\{0,1,2,\ldots\}). \tag{11}$$

Let us now consider more carefully the system of equations in moments (11). From this system changing $k = \overline{1, N}$ and neglecting the moments, order of which is more than N, we obtain a system of equations of Nth approximation. Giving N different values (starting from zero) we obtain the systems of equations of various approximation for prismatic bodies of constant thickness. It should be noted that with the method of deriving a system of equations in moments of displacement vector considered below with respect to a system of orthogonal polynomials, the resulting system of equations in moments, the order of approximation of which is greater than zero, is divided into two systems, one of which contains moments of even order of the unknown displacement vector, and the other contains moments of odd order. In addition, the resulting systems with even and odd moments of the displacement vector, in turn, are easily split separately and for each moment of the displacement vector an elliptic equation of higher order is obtained, for which, using the Vekua method, a presentation of an analytical solution can be obtained (see in [39]). Below, in the case of quasi-statics, equations of the tenth order of approximation are obtained in the moments of the displacement vector with respect to the system of Legendre polynomials and everything said above is demonstrated in this example.

1.2.3 Quasi-static equations of second strain gradient theory of isotropic prismatic bodies of constant thickness in moments of displacement vector with respect to the system of Legendre polynomials

To obtain the desired systems of equations we need to find expression for the so-called "stroke-operators" $\overset{(k)}{\mathbf{u}}''$, $\overset{(k)}{\mathbf{u}}^{IV}$, $\overset{(k)}{\mathbf{u}}^{VI}$ and $\overset{(k)}{\mathbf{u}}^{VIII}$ when $-1 \le x^3 \le 1$. Note that in the application of the system of Legendre polynomials, the expressions for the stroke-operators $\overset{(k)}{\mathbf{u}}''$, $\overset{(k)}{\mathbf{u}}^{IV}$, $\overset{(k)}{\mathbf{u}}^{VI}$, $\overset{(k)}{\mathbf{u}}^{VIII}$

and so on are defined using the following relationship (see in [35]):

$$\begin{aligned}
&\overset{(n)}{\mathbf{u}}{}^{(2m)}(x') = (2n+1)\sum_{k=1}^{\infty} C_{k+2m-2}^{2m-1}\prod_{s=1}^{2m-1}(2n+2k+2s-1)\overset{(n+2k+2m-2)}{\mathbf{u}}\\
&= \frac{2n+1}{2}\sum_{k=1}^{2m}(-1)^{k+1}[(\partial_3^{2m-k}\mathbf{u})^+ + (-1)^{n+k}(\partial_3^{2m-k}\mathbf{u})^-]P_n^{k-1}(1) + \overset{(k)}{\underline{\mathbf{u}}}{}^{(2m)},\\
&\overset{(k)}{\underline{\mathbf{u}}}{}^{(2m)} = (2n+1)\sum_{k=1}^{[n/2-m+1]} C_{k+2m-2}^{2m-1}\prod_{s=1}^{2m-1}(2n-2k-2s+3)\overset{(n-2k-2m+2)}{\mathbf{u}},\\
&n\in\mathbb{N}_0,\ m\in\mathbb{N}.
\end{aligned} \tag{12}$$

Here $\mathbb{N}_0$ is the set of non-negative integers, but $\mathbb{N}$ is the set of natural numbers, C_{k+2m-2}^{2m-1} are binomial coefficients, $(\partial_3^s\mathbf{u})^- = (\partial_3^s\mathbf{u})\big|_{x^3=-1}$ and $(\partial_3^s\mathbf{u})^+ = (\partial_3^s\mathbf{u})\big|_{x^3=1}$, $s\in\mathbb{N}_0$.

It is seen from (12) that the stroke-operators $\overset{(k)}{\mathbf{u}}{}''$, $\overset{(k)}{\mathbf{u}}{}^{IV}$, $\overset{(k)}{\mathbf{u}}{}^{VI}$, $\overset{(k)}{\mathbf{u}}{}^{VIII}$ and so on are represented as an infinite sum of moments of displacement vector, or as the final sum of moments of displacement vector and the sum of values of displacement vector and their partial derivatives in x^3 on the face surfaces, i.e. when $x^3=-1$ and $x^3=1$.Therefore, taking into account (12), from (11) we get different representations of the systems of equations of the quasi-static second strain gradient theory of isotropic prismatic thin bodies of constant thickness in moments of displacement vector with respect to the system of Legendre polynomials.

Let us write the expressions for some stroke-operators $\overset{(k)}{\mathbf{u}}{}''$, $\overset{(k)}{\mathbf{u}}{}^{IV}$, $\overset{(k)}{\mathbf{u}}{}^{VI}$ and $\overset{(k)}{\mathbf{u}}{}^{VIII}$ when $-1\le x^3\le 1$, which are obtained based on the first equality in (12). By virtue of simple calculations when considering, for example, the tenth order approximation, we obtain

$$\begin{aligned}
&\overset{(0)}{\mathbf{u}}{}'' = 3\overset{(2)}{\mathbf{u}} + 10\overset{(4)}{\mathbf{u}} + 21\overset{(6)}{\mathbf{u}} + 36\overset{(8)}{\mathbf{u}} + 55\overset{(10)}{\mathbf{u}},\\
&\overset{(0)}{\mathbf{u}}{}^{IV} = 105\overset{(4)}{\mathbf{u}} + 1260\overset{(6)}{\mathbf{u}} + 6930\overset{(8)}{\mathbf{u}} + 25740\overset{(10)}{\mathbf{u}},\\
&\overset{(0)}{\mathbf{u}}{}^{VI} = 10395\overset{(6)}{\mathbf{u}} + 270270\overset{(8)}{\mathbf{u}} + 2837835\overset{(10)}{\mathbf{u}},\\
&\overset{(0)}{\mathbf{u}}{}^{VIII} = 2027025\overset{(8)}{\mathbf{u}} + 91891800\overset{(10)}{\mathbf{u}};\\
&\overset{(1)}{\mathbf{u}}{}'' = 15\overset{(3)}{\mathbf{u}} + 42\overset{(5)}{\mathbf{u}} + 81\overset{(7}{\mathbf{u}} + 132\overset{(9)}{\mathbf{u}},\\
&\overset{(1)}{\mathbf{u}}{}^{IV} = 945\overset{(5)}{\mathbf{u}} + 8316\overset{(7)}{\mathbf{u}} + 38610\overset{(9)}{\mathbf{u}},\\
&\overset{(1)}{\mathbf{u}}{}^{VI} = 135135\overset{(7)}{\mathbf{u}} + 2432430\overset{(9)}{\mathbf{u}},\\
&\overset{(1)}{\mathbf{u}}{}^{VIII} = 34459425\overset{(9)}{\mathbf{u}};\\
&\overset{(2)}{\mathbf{u}}{}'' = 35\overset{(4)}{\mathbf{u}} + 90\overset{(6)}{\mathbf{u}} + 165\overset{(8)}{\mathbf{u}} + 260\overset{(10)}{\mathbf{u}},\\
&\overset{(2)}{\mathbf{u}}{}^{IV} = 3465\overset{(6)}{\mathbf{u}} + 25740\overset{(8)}{\mathbf{u}} + 107250\overset{(10)}{\mathbf{u}},\\
&\overset{(2)}{\mathbf{u}}{}^{VI} = 675675\overset{(8)}{\mathbf{u}} + 9845550\overset{(10)}{\mathbf{u}},\\
&\overset{(2)}{\mathbf{u}}{}^{VIII} = 218243025\overset{(10)}{\mathbf{u}};\ldots
\end{aligned} \tag{13}$$

In the case under consideration, the displacement vector is represented as

$$\mathbf{u}(x^1,x^2,x^3) = \overset{(0)}{\mathbf{u}}(x^1,x^2) + \overset{(1)}{\mathbf{u}}(x^1,x^2)P_1(x^3) + \cdots + \overset{(10)}{\mathbf{u}}(x^1,x^2)P_{10}(x^3),$$

where $P_k(x^3)$, $k=\overline{1,10}$ are Legendre polynomials.

1.2.4 *System of the quasi-static equations of the second strain gradient theory of homogeneous elastic isotropic prismatic bodies of constant thickness in the moments of the displacement vector of odd and even orders with respect to the Legendre polynomials of the tenth order approximation*

As mentioned above, by virtue of the method of orthogonal polynomials, a system of equations in moments is obtained, which is split into two systems of equations, one of which contains moments of even order of the unknown displacement vector, and the other contains moments of odd order. In this case, the resulting systems with even and odd moments of the displacement vector are separately split and for each moment of the displacement vector an elliptic-type equation of higher order is obtained, for which a presentation of an analytical solution can be obtained by the Vekua method. In fact, it is easy to see that due to (12) (see also (13)) from (11) we obtain a system of equations with odd and even moments of the displacement vector with respect to the system of Legendre polynomials the tenth order approximation in the form

$$
\begin{aligned}
&(l_2^4\bar\Delta^2 - l_1^2\bar\Delta + 1)\bar\Delta^2\overset{(0)}{\mathbf{u}} + 3h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta\overset{(2)}{\mathbf{u}}\\
&+[10h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta + 105h^{-4}(6l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta - 1)]\overset{(4)}{\mathbf{u}}\\
&+[21h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta + 1260h^{-4}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\\
&+10395h^{-6}(4l_2^4\bar\Delta - l_1^2)]\overset{(6)}{\mathbf{u}} + [36h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta\\
&+6930h^{-4}(6l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta - 1) + 270270h^{-6}(4l_2^4\bar\Delta - l_1^2) + 4054050h^{-8}l_2^4]\overset{(8)}{\mathbf{u}}\\
&+[55h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta + 25740h^{-4}(6l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta - 1)\\
&+2837835h^{-6}(4l_2^4\bar\Delta - l_1^2) + 183783600h^{-8}l_2^4]\overset{(10)}{\mathbf{u}} + \overset{(0)}{G} = 0,\\
&(l_2^4\bar\Delta^2 - l_1^2\bar\Delta + 1)\bar\Delta^2\overset{(1)}{\mathbf{u}} + 15h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta\overset{(3)}{\mathbf{u}}\\
&+[42h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2) + 945h^{-4}(6l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta - 1)]\bar\Delta\overset{(5)}{\mathbf{u}}\\
&+[81h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta + 8316h^{-4}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\\
&+135135h^{-6}(4l_2^4\bar\Delta - l_1^2)]\overset{(7)}{\mathbf{u}} + [132h^{-2}(4l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta + 2)\bar\Delta\\
&+38610h^{-4}(6l_2^4\bar\Delta^2 - 3l_1^2\bar\Delta - 1) + 2432430h^{-6}(4l_2^4\bar\Delta - l_1^2)\\
&+68918850h^{-8}l_2^4]\overset{(9)}{\mathbf{u}} + \overset{(1)}{G} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(2)}{\mathbf{u}} + 35h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(4)}{\mathbf{u}}\\
&+[90h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 3465h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\overset{(6)}{\mathbf{u}}\\
&+[165h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 25740h^{-4}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\\
&+75675h^{-6}(4l_2^4\bar{\Delta} - l_1^2)]\overset{(8)}{\mathbf{u}} + [260h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\\
&+107250h^{-4}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2) + 9845550h^{-6}(4l_2^2\bar{\Delta} - l_1^2)\\
&+436486050h^{-8}l_2^4]\overset{(10)}{\mathbf{u}} + \overset{(2)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(3)}{\mathbf{u}} + 63h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(5)}{\mathbf{u}}\\
&+[154h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 9009h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\overset{(7)}{\mathbf{u}}\\
&+[273h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 60060h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)\\
&+2297255h^{-6}(4l_2^4\bar{\Delta} - l_1^2)]\overset{(9)}{\mathbf{u}} + \overset{(3)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(4)}{\mathbf{u}} + 99h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(6)}{\mathbf{u}}\\
&+[234h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 19305h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\overset{(8)}{\mathbf{u}}\\
&+[405h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 119340h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)\\
&+6235715h^{-6}(4l_2^2\bar{\Delta} - l_1^2)]\overset{(10)}{\mathbf{u}} + \overset{(4)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(5)}{\mathbf{u}} + 143h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(7)}{\mathbf{u}}\\
&+[330h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\\
&+36465h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\overset{(9)}{\mathbf{u}} + \overset{(5)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(6)}{\mathbf{u}} + 195h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(8)}{\mathbf{u}}\\
&+[442h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\\
&+62985h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\overset{(10)}{\mathbf{u}} + \overset{(6)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(7)}{\mathbf{u}} + 255h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(9)}{\mathbf{u}} + \overset{(7)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(8)}{\mathbf{u}} + 323h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2 + 2)\bar{\Delta}\overset{(10)}{u} + \overset{(8)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(9)}{\mathbf{u}} + \overset{(9)}{G} = 0,\ \ (l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(10)}{\mathbf{u}} + \overset{(10)}{G} = 0.
\end{aligned}
\tag{14}
$$

Note that from the system of equations of the tenth approximation, systems of equations of lower approximation (ninth, eighth, seventh, etc.) can be easily obtained. For example, we get the system of equations of the ninth approximation if we remove the last equation from the system (14), and remove the terms containing moments of the tenth order from the rest of the equations.

Let us separate the system of equations containing odd moments from the system of equations (14). We Have

$$
\begin{aligned}
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(1)}{\mathbf{u}} + 15h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(3)}{\mathbf{u}} + [42h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\\
&+945h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\bar{\Delta}\overset{(5)}{\mathbf{u}} + [81h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\\
&+8316h^{-4}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2) + 135135h^{-6}(4l_2^4\bar{\Delta} - l_1^2)]\overset{(7)}{\mathbf{u}} + [132h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\\
&+38610h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1) + 2432430h^{-6}(4l_2^4\bar{\Delta} - l_1^2) + 68918850h^{-8}l_2^4]\overset{(9)}{\mathbf{u}} + \overset{(1)}{G} = 0,\\
&\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(3)}{\mathbf{u}} + 63h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(5)}{\mathbf{u}}\\
&+[154h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 9009h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\overset{(7)}{\mathbf{u}}\\
&+[273h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 60060h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)\\
&+2297255h^{-6}(4l_2^4\bar{\Delta} - l_1^2)]\overset{(9)}{\mathbf{u}} + \overset{(3)}{G} = 0,\\
&\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(5)}{\mathbf{u}} + 143h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(7)}{\mathbf{u}}\\
&+[330h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta} + 36465h^{-4}(6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1)]\overset{(9)}{\mathbf{u}} + \overset{(5)}{G} = 0,\\
&\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(7)}{\mathbf{u}} + 255h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta}\overset{(9)}{\mathbf{u}} + \overset{(7)}{G} = 0,\\
&(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2\overset{(9)}{\mathbf{u}} + \overset{(9)}{G} = 0.
\end{aligned}
\tag{15}
$$

We write this system in a matrix form. For this purpose, we introduce the following designations:

$$
L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ 0 & L_{22} & L_{23} & L_{24} & L_{25} \\ 0 & 0 & L_{33} & L_{34} & L_{35} \\ 0 & 0 & 0 & L_{44} & L_{45} \\ 0 & 0 & 0 & 0 & L_{55} \end{pmatrix}, \ U = \begin{pmatrix} \overset{(1)}{u} \\ \overset{(3)}{u} \\ \overset{(5)}{u} \\ \overset{(7)}{u} \\ \overset{(9)}{u} \end{pmatrix}, \ G = \begin{pmatrix} \overset{(1)}{G} \\ \overset{(3)}{G} \\ \overset{(5)}{G} \\ \overset{(7)}{G} \\ \overset{(9)}{G} \end{pmatrix}, \tag{16}
$$

where

$$
L_{11} = L_{22} = L_{33} = L_{44} = L_{55} = (l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)\bar{\Delta}^2, \quad L_{12} = 15h^{-2}(4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2)\bar{\Delta},
$$

$$L_{13}=42h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta}+945h^{-4}(6l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}-1),$$

$$L_{14}=81h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta}+8316h^{-4}(6l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}-1)+10384h^{-6}(4l_2^4\bar{\Delta}-l_1^2),$$

$$L_{15}=132h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta}+386106h^{-4}(6l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}-1)$$

$$+2432430h^{-6}(4l_2^4\bar{\Delta}-l_1^2)+68918850h^{-8}l_2^4,\ \ L_{23}=63h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta},$$

$$L_{24}=154h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta}+9009h^{-4}(6l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}-1),$$

$$L_{25}=154h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta}+9009h^{-4}(6l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}-1),$$

$$L_{34}=143h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta},\ \ L_{45}=255h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta},$$

$$L_{35}=330h^{-2}(4l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}+2)\bar{\Delta}+36465h^{-4}(6l_2^4\bar{\Delta}^2-3l_1^2\bar{\Delta}-1).$$

Then, the system (15) shall be written in a matrix form:

$$LU+G=0 \tag{17}$$

Denoting by L_* the matrix of cofactors for the differential matrix L (see (16)), we will have

$$L_*=\begin{pmatrix} L_{*11} & 0 & 0 & 0 & 0\\ L_{*21} & L_{*22} & 0 & 0 & 0\\ L_{*31} & L_{*32} & L_{*33} & 0 & 0\\ L_{*41} & L_{*42} & L_{*43} & L_{*44} & 0\\ L_{*51} & L_{*52} & L_{*53} & L_{*54} & L_{*55} \end{pmatrix}, \tag{18}$$

It is easy to find expressions for the elements of the matrix of cofactors L_* (see (18)) and its determinant $|L_*|$. Indeed, due to simple, but labor-intensive calculations we will have

$$L_{*11}=L_{*22}=L_{*33}=L_{*44}=L_{*55}=(l_2^4\bar{\Delta}^2-l_1^2\bar{\Delta}+1)^4\bar{\Delta}^8,$$

$$|L_*|=(l_2^4\bar{\Delta}^2-l_1^2\bar{\Delta}+1)^{20}\bar{\Delta}^{40},$$

$$L_{*21}=-15h^{-2}b_0^3b_1\bar{\Delta}^7,\ \ L_{*31}=(945h^{-4}b_0^2b_1^2-a_0b_0^3)\bar{\Delta}^6,$$

$$L_{*32}=-63h^{-2}b_0^3b_1\bar{\Delta}^7,$$

$$L_{*41}=-\bar{\Delta}^5(135135h^{-6}b_0^2b_1^3+a_1\bar{\Delta}b_0^3-143h^{-2}a_0b_0^2b_1-15h^{-2}c_0b_0^2b_1),$$

$$L_{*42}=\bar{\Delta}^6b_0^2(9009h^{-4}b_1^2-b_0c_0),\ \ L_{*43}=-143h^{-2}\bar{\Delta}^7b_0^3b_1,$$

$$L_{*51}=\bar{\Delta}^4(34459425h^{-8}b_1^4-a_2\bar{\Delta}^4b_0^3-36465h^{-4}a_0b_0^2b_1^2+255h$$

$$-2a_1b_0^2b_1-3825h^{-2}c_0b_1^2b_0-a_0b_0^2d_0+15h-2c_1b_0^2b_1-945h^{-4}d_0b_1^2b_0),$$

$$L_{*52}=-\bar{\Delta}^5(32297295h^{-6}b_0b_1^3+\bar{\Delta}^4b_0^3c_1-255h^{-2}c_0b_0^2b_1-63h^{-2}d_0b_0^2b_1)$$

$$L_{*53}=\bar{\Delta}^6b_0^2(36465h^{-4}b_1^2-d_0b_0),\ \ L_{*54}=-\bar{\Delta}^6b_0^3(255h^{-2}b_1\bar{\Delta}-d_0).$$

$$b_0 = l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1,\ \ b_1 = 4l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} + 2,$$

$$b_2 = 6l_2^4\bar{\Delta}^2 - 3l_1^2\bar{\Delta} - 1,\ \ b_3 = 4l_2^4\bar{\Delta}^2 - l_1^2,$$

$$a_0 = 42h^{-2}b_1\bar{\Delta} + 945h^{-4}b_2,$$

$$a_1 = 81h^{-2}b_1\bar{\Delta} + 8316h^{-4}b_2 + 10384h^{-6}b_3,$$

$$a_2 = 132h^{-2}b_1\bar{\Delta}+386106h^{-4}b_2+2432430h^{-6}b_3+68918850h^{-8}l_2^4,$$

$$c_1=154h^{-2}b_1\bar{\Delta}+9009h^{-4}b_2,$$

$$c_0 = 273h^{-2}b_1\bar{\Delta} + 60060h^{-4}b_2 + 2297255h^{-6}b_3,$$

$$d_0 = 330h^{-2}b_1\bar{\Delta} + 36465h^{-4}b_2.$$

It is easy to see that the matrix of cofactors L_* (18) can be represented in the following form:

$$L_* = \bar{\Delta}^4 N, \quad N = \begin{pmatrix} N_{11} & 0 & 0 & 0 & 0 \\ N_{21} & N_{22} & 0 & 0 & 0 \\ N_{31} & N_{32} & N_{33} & 0 & 0 \\ N_{41} & N_{42} & N_{43} & N_{44} & 0 \\ N_{51} & N_{52} & N_{53} & N_{54} & N_{55} \end{pmatrix}, \tag{19}$$

where

$$N_{11}=N_{22}=N_{33}=N_{44}=L_{55}=(l_2^4\bar{\Delta}^2-l_1^2\bar{\Delta}+1)^4\bar{\Delta}^4,\ \ N_{21} = -15h^{-2}b_0^3b_1\bar{\Delta}^3,$$

$$N_{31} = (945h^{-4}b_0^2b_1^2 - a_0b_0^3)\bar{\Delta}^2,\ \ N_{32} = -63h^{-2}b_0^3b_1\bar{\Delta}^3,$$

$$N_{41} = -\bar{\Delta}(135135h^{-6}b_0^2b_1^3 + a_1\bar{\Delta}b_0^3 - 143h^{-2}a_0b_0^2b_1 - 15h^{-2}c_0b_0^2b_1),$$

$$N_{42} = \bar{\Delta}^2b_0^2(9009h^{-4}b_1^2 - b_0c_0),\ \ N_{43} = -143h^{-2}\bar{\Delta}^3b_0^3b_1,$$

$$N_{51} = (34459425h^{-8}b_1^4 - a_2\bar{\Delta}^4b_0^3 - 36465h^{-4}a_0b_0^2b_1^2 + 255h{-}2a_1b_0^2b_1$$

$$-3825h^{-2}c_0b_1^2b_0 - a_0b_0^2d_0 + 15h{-}2c_1b_0^2b_1 - 945h^{-4}d_0b_1^2b_0),$$

$$N_{52} = -\bar{\Delta}(32297295h^{-6}b_0b_1^3 + \bar{\Delta}^4b_0^3c_1 - 255h^{-2}c_0b_0^2b_1 - 63h^{-2}d_0b_0^2b_1),$$

$$N_{53} = \bar{\Delta}^2b_0^2(36465h^{-4}b_1^2 - d_0b_0),\ \ \ L_{*54} = -\bar{\Delta}^2b_0^3(255h^{-2}b_1\bar{\Delta} - d_0).$$

Applying to the equation (17) on the left the matrix operator N^T and taking into account the equality

$$N^TL = E(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)^5\bar{\Delta}^6,$$

where E is the identity matrix of the fifth order, we obtain

$$(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)^5\bar{\Delta}^6U + N^TG = 0,$$

or

$$(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)^5\bar{\Delta}^6\overset{(2k+1)}{\mathbf{u}} + \overset{(2k+1)}{\mathbf{\Phi}} = 0,\ \ k = 0, 1, 2, 3, 4. \tag{20}$$

In the absence of body forces the equation (20) takes the following form:

$$(l_2^4\bar{\Delta}^2 - l_1^2\bar{\Delta} + 1)^5\bar{\Delta}^6 \overset{(2k+1)}{\mathbf{u}} = 0, \;\; k = 0,1,2,3,4. \tag{21}$$

The system of equations of the even moments shall also be considered similarly. In order to shorten the letter, we will not dwell on this.

Let us write the presentation of the analytical solution of such types of equations (see (21)) by I. Vekua's method, which is presented in his monograph [39]. To do this, let's rewrite equation (21) as follows:

$$\bar{\Delta}^6(\bar{\Delta} + \varkappa_1)^5(\bar{\Delta} + \varkappa_2)^5 \overset{(2k+1)}{\mathbf{u}} = 0, \;\; k = 0,1,2,3,4,$$

where $\varkappa_1$ and $\varkappa_2$ are the roots of the equation

$$l_2^4\varkappa^2 - l_1^2\varkappa + 1 = 0, \;\; \varkappa_{1,2} = \frac{l_1^2 \pm \sqrt{l_1^4 - 4l_2^4}}{2l_2^2}.$$

Assume that $\varkappa_1 \neq \varkappa_2$. Then all solutions (the general solution) of the equations (21) are obtained from the formulas

$$\overset{(2k+1)}{\mathbf{u}} = \overset{(2k+1)}{\mathbf{v}} + \sum_{j=1}^{2}\sum_{l=0}^{4} r^l \frac{\partial^l \overset{(2k+1)}{\mathbf{u}}_{jl}}{\partial r^l}, \;\; k = 0,1,2,3,4, \text{ respectively}, \;\; r = |z|,$$

where $z = x + iy$ is a complex variable, $\overset{(2k+1)}{\mathbf{u}}_{jl}$, $k = 0,1,2,3,4$, are any solutions of the equations

$$(\bar{\Delta} + \varkappa_j)\overset{(2k+1)}{\mathbf{u}} = 0, \;\; k = 0,1,2,3,4, \text{ respectively},$$

$\overset{(2k+1)}{\mathbf{v}}$, $k = 0,1,2,3,4$, are solutions of the equations $\bar{\Delta}^6\overset{(2k+1)}{\mathbf{v}} = 0$, $k = 0,1,2,3,4$, respectively and the functions $\overset{(2k+1)}{\mathbf{v}}$, $\overset{(2k+1)}{\mathbf{u}}_{jl}$ are uniquely determined by $\overset{(2k+1)}{\mathbf{u}}$.

2 Conclusions and future steps

Vector motion equation of the second strain and velocity gradient linear elasticity theory for the displacement vector for an arbitrary anisotropic body is obtained. This equation depends on both scaling tensors for statics and dynamics. From this equation motion equation of the second strain and velocity gradient linear theory for isotropic homogeneous elastic media is derived, from which the decomposed motion equation with respect to the displacement vector is given. From the decomposed motion equation the quasi-static equation of the second strain gradient of the linear theory of thin prismatic bodies with respect to the displacement vector is derived under the classical parameterization of the body domain. From the last quasi-static equation, the infinite system of equations in the moments of the displacement vector with respect to an arbitrary system of orthogonal polynomials is obtained. From this system of equations the system of equations in the moments of the displacement vector with respect to the system of Legendre polynomials of several first approximations (from the zeroth to the tenth approximation) are derived. This system, analogous to any system with any order of approximation, splits into two systems, one of which contains the moments of the displacement vector of an even order, and the other contains the moments of an odd order. In this paper the system, containing the moments of the displacement vector of an odd order, is decomposed and for each moment included in this system of equations, a separate equation of elliptic type of higher order is obtained, for which a representation of the analytical solution can be written using the Vekua method (see [39]). In this work, a representation of the analytical solution of one higher-order elliptic-type equation is given.

It should also be noted that all the above can be easily extended to the theories of other rheological bodies, including higher-order gradient theories, as well as to linear and nonlinear classical and micropolar gradient theories of elastic and viscoelastic bodies, which represent problems for further research.

3 Conflict of interest

The author declares no conflicts of interest.

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