A Matrix Laplacian for Fractal Strings: Generalised Trigonometric Functions as Limits of Recurrence Relations II

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In the first part of this paper [1], an expression for a spectrum encoding function for standard fractal strings [2] was given. For their measure theoretic counterparts (see for example [3]), Arzt recursively defined in [4] a generalised trigonometric function encoding their spectra. For a subset of these strings where the mass matrix is simply a multiple of the identity matrix, we obtain another recurrence relation providing a more efficient way to approximate the eigenvalues of the Dirichlet Laplacian. The spectrum encoding function at approximation level n is then:

$$p_n(\lambda) = A_n p_{n-1}(\lambda) + B_n \sum_{i=0}^{n-2} \prod_{k=i+2}^{n-1} D_k C_{i+1} p_i(\lambda) + (\lambda - 2) B_n \prod_{i=1}^{n-1} D_i,$$

with:

$$A_i = \kappa^{-4}\lambda^4 + \left(-6 - 2r_i^{-1}\right)\kappa^{-3}\lambda^3 + \left(11 + 10r_i^{-1}\right)\kappa^{-2}\lambda^2 + \left(-7 - 13r_i^{-1}\right)\kappa^{-1}\lambda + \left(1 + 4r_i^{-1}\right),$$

$$B_i = -\kappa^{-3}\lambda^3 + \left(5 + r_i^{-1}\right)\kappa^{-2}\lambda^2 + \left(-6 - 4r_i^{-1}\right)\kappa^{-1}\lambda + \left(1 + 3r_i^{-1}\right),$$

$$C_i = \kappa^{-3}\lambda^3 + \left(-4 - 2r_i^{-1}\right)\kappa^{-2}\lambda^2 + \left(4 + 6r_i^{-1}\right)\kappa^{-1}\lambda + \left(-1 - 3r_i^{-1}\right),$$

$$D_i = -\kappa^{-2}\lambda^2 + \left(3 + r_i^{-1}\right)\kappa^{-1}\lambda + \left(-1 - 2r_i^{-1}\right),$$

where $p_0(\lambda) = \kappa^{-2}\lambda^2 - 4\kappa^{-1}\lambda + 3$ and the r_i describe the underlying fractal structure. As an example, for the measure theoretic triadic Cantor string $\kappa = 2 \cdot 6^{j+1}$ for $n = 2^j - 1$, $j = 0, 1, 2, \dots$ and the r_i are given by ($\lfloor \cdot \rfloor$ denoting the floor function):

$$r_i = 3^{-\sum_{k=2}^{\infty} \left\lfloor \frac{2^k \left\lfloor \frac{i+1}{2^{k-2}} \right\rfloor}{(4i+4)} \right\rfloor}.$$

References

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