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ON AN EXPLICIT CONSTRUCTION OF BICENTRIC QUADRILATERALS

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Abstract. We present an effective method of constructing bicentric quadrilaterals. More precisely, we give an explicit construction of bicentric quadrilateral with prescribed two sides. In the symmetrical case, the distance between the two centers of the arising bicentric quadrilateral is computed, which in virtue of Fuss relation also gives an explicit formula for the radius of the corresponding incircle. An analogous construction and some of its properties are given for bicentric polygons with an arbitrary number of sides. In conclusion we present an interpretation of the main results in terms of the Kendall shape space and numerical results in a concrete case.

Keywords and phrases: Cyclic polygon, tangential polygon, bicentric polygon, kite, Fuss relations, Kendall shape space.

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1. Introduction

We begin with a few definitions and notations. In this paper, we only deal with the so-called planar polygons lying in a given Euclidean plane which is endowed with a fixed Cartesian coordinate system. Hence we can speak of distances, areas and other metric characteristics of polygons in question.

Definition 1. A (planar) polygon which has a circumscribed circle, i.e. all of its vertices belong to a certain circle, is called a cyclic polygon.

Definition 2. A (planar) polygon which has an inscribed circle, i.e. a circle that is tangent to each side of the polygon, is called a tangential polygon.

Definition 3. A (planar) polygon which simultaneously has a circumscribed circle and an inscribed circle, is called a bicentric polygon.

In other words, a polygon P is bicentric if it is simultaneously cyclic and tangential. For a bicentric polygon we always denote by $R = R(P)$ (circumradius) and $r = r(P)$ (inradius) the radii of circumscribed and inscribed circles respectively. The distance between the centers of these two circles will be called the eccentricity $d(P)$ of P. If a bicentric polygon P is fixed and no misunderstanding can arise, we write simply (R, r, d) and call it the Euler triple of P. Obviously, each triangle is bicentric and each regular polygon is bicentric with eccentricity equal to zero.

As it is known from the classical geometry, a polygon with more than three sides in general is not bicentric. This only happens under special conditions on the shape of polygon. For a bicentric polygon, there are important relations between the inradius, circumradius and eccentricity, which may be considered as necessary conditions of bicentricity.

The first such relation was given by L.Euler. Namely, for a triangle T one has

$$R^2 - d^2 = 2Rr,$$

where (R, r, d) is the Euler triple of T introduced above.

The first one who was concerned with general bicentric polygons was the German mathematician Nicolaus Fuss. He found analogous relations (necessary conditions) for bicentric quadrilaterals, pentagons, hexagons, heptagons and octagons. For this reason, those relations are known as Fuss's relations, also in the cases where $n > 8$.

A very remarkable theorem concerning bicentric polygons was given by the French mathematician Victor Poncelet. This theorem can be stated as follows. If there is one bicentric n-gon whose circumcircle is C_1 and incircle is C_2, then there are infinitely many bicentric n-gons whose circumcircle is C_1 and incircle is C_2. This famous theorem dates to the nineteenth century. Since then many mathematicians worked on the problems connected with this theorem and solved many of them.

In this paper, we present an effective method of constructing bicentric polygons and establish certain properties of this construction.

2. Two lemmas and Fuss relation

We now formulate two auxiliary lemmas and recall Fuss relation for quadrilaterals used in the sequel.

Lemma 1. *Let us suppose that we have any two circumferences C_2 and C_1, where R and r are, respectively, their radii, C_1 is completely inside C_2, the distance between the two centers is denoted d. Then, if we draw any tangent line to circle C_1, divided by the point of tangency into two segments t_1 and t_2, and t_1 will be given, we can calculate t_2 as a function of t_1 and the relation has the following form*

$$(t_2)_{1,2} = \frac{(R^2 - d^2)t_1 \pm \sqrt{D}}{(t_1^2 + r^2)},$$

where

$$D = t_1^2(R^2 - d^2)^2 + (r^2 + t_1^2)[4R^2d^2 - r^2t_1^2 - (R^2 + d^2 - r^2)^2].$$

Lemma 2. *Suppose we have a triangle ABC inscribed in a circumference of radius R. Given the two sides u, v of this triangle one can compute its third side w as follows:*

$$w = \frac{1}{2}\frac{\sqrt{4R^2u^2 + 4R^2v^2 - 2u^2v^2 \pm 2\sqrt{D}}}{R},$$

where

$$D = 16R^4u^2v^2 - 4R^2u^4v^2 - 4R^2u^2v^4 + u^4v^4.$$

Finally, recall that the classical Fuss relation for quadrilaterals has the form

$$(R^2 - d^2)^2 - 2r^2(R^2 + d^2) = 0, \tag{1}$$

where (R, r, d) is the Euler triple introduced above. Now we are ready to present our main results.

3. The main results

Our first main result refers to bicentric quadrilaterals.

Theorem 1. *For a triple of points (A, B, C) in a circumference S with $|AB| = a > 0$, $|BC| = b > 0$, there exists the fourth point D in S such that the quadrilateral $< A, B, C, D >$ is bicentric.*

Proof. To prove this let us consider the bisector L of the angle formed by AB and BC sides and a point T on this bisector. We wish to choose T in such a way that it becomes the center of the inscribed circle X of the sought bicentric quadrilateral so that the sides AB and BC are tangent to X. To this end we move the point T along L from the position $T = B$ and consider the uniquely defined circle $X(T)$ with the center at T which is tangent to the given sides. Then let us draw the two tangent lines to $X(T)$ from the points A and C and denote by A' and C' the second points of intersection of those tangent lines with the circle S. It is now easy to see and verify by computer experiments that, as we move the point T along L, the points A' and C' monotonically move towards each other along the circle S. So for a certain position of T the points A' and C' will coincide and this point is exactly the fourth vertex of the sought bicentric quadrilateral. It is easy to make this argument completely rigorous so that the proof is complete.

For brevity and convenience, the bicentric quadrilateral $< A, B, C, D >$ given by this theorem will be called the bicentric closure $Q(A, B, C)$ of the points A, B, C. To compute the position of point D and the Euler triple of the arising bicentric quadrilateral in general is not an easy problem but we have an explicit formula for its eccentricity in a special case where $< A, B, C, D >$ is a kite.

Theorem 2. *For a triple of points (A, B, C) in a circumference S with $|AB| = |BC| = a > 0$, the eccentricity of the unique bicentric closure $Q(A, B, C)$ is given by the formula*

$$|OO_t| = d = \frac{R\left(\sqrt{4\,R^2 - a^2} + a\right)}{\sqrt{4\,R^2 - a^2} - a} \tag{2}$$

Proof. We consider a circumference S of radius R with the center at $O(0, 0)$ and the points on this circle with coordinates: $A(x_1, y_1)$, $B(x_2, y_2) = (0, R)$, $C(x_3, y_3)$, $D(x_4, y_4) = (0, -R)$. Next we introduce the notation $|AB| = |BC| = a$, $|AD| = |CD| = k$, where

$$k = \sqrt{4R^2 - a^2},$$
$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$
$$|AD| = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2}.$$

Our goal is to compute the distance between the incenter and circumcenter of $< A, B, C, D >$ through the coordinates of points A, B, C, D and the radius R of the circle S. As above we denote the distance between the two centers by d and denote the center of the incircle by O_t with coordinates $(0, y_t)$.

To calculate the coordinate y_t we use the formula for calculating the distance between a point and a line from analytical geometry which gives

$$dist(O_t, > AB <) = \frac{-x_1\,(y_1 - y_t) - x_1\,(R - y_1)}{a}, \tag{3}$$

$$dist(O_t, > AD <) = \frac{x_1\,(y_t + R)}{k}, \tag{4}$$

Taking into account that expressions (3) and (4) should be equal and solving the arising equation with respect to y_t, we get:

$$y_t = \frac{R\,(k + a)}{k - a}. \tag{5}$$

Finally, we get

$$|OO_t| = d = \frac{R\left(\sqrt{4\,R^2 - a^2} + a\right)}{\sqrt{4\,R^2 - a^2} - a}$$

as it was stated. The proof is complete.

Using the Fuss relation we can also compute the inradius of $< A, B, C, D >$.

Theorem 3. *For a triple of points (A, B, C) in a circumference S with $|AB| = |BC| = a > 0$, the inradius of the unique bicentric closure $Q(A, B, C)$ is given by the formula*

$$r = \frac{1}{2}\left(\sqrt{(4R^2 - 2a\sqrt{4R^2 - a^2})(4R^2 - a^2)a}\right) / \left(2R^2 - a\sqrt{4R^2 - a^2}\right)$$

Proof. We use Fuss relation (1) and equation (2). Namely, we insert (2) into (1) and calculate r, which gives the above formula for the inradius.

4. Computing the bicentric closure in the general case

As it was mentioned, in the general case, where a is not equal to b, it is not easy to compute the coordinates of D and the Euler triple of $< A, B, C, D >$. To obtain a virtual solution of this problem we use another method of constructing a bicentric closure using analytical geometry.

Given points A, B, C on a circle S of radius is R and center at $O(0, 0)$, we denote the coordinates of given points by $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and the coordinates of the sought point by $D(x, y)$. The following observation follows from the definition of hyperbola and our assumptions.

Lemma 3. *If we rotate the coordinate system by an angle α so that the center O moves to point O_1 and points A and C will be located on the rotated x axis and construct a hyperbola so that the points A and C will be its foci, then at the intersection of the hyperbola and the circle C_1 will be the fourth point D. Hence to calculate the coordinates of D, it is enough to solve the system, consisting of the hyperbola equation and circle equations. After doing some calculations, we get the following system of equations:*

Equation 1.

$$\frac{u^2}{a^2} - \frac{v^2}{-a^2+c^2} = 2\frac{(-2xR^2+2xRx_3+2yp+(R+x_3)q)^2}{k+m+n+w}$$
$$+3\,\frac{\left(2\,yR^2-2\,yRx_3-y_3q+2\,xp\right)^2}{k+4\,m+4\,n+4\,w} = 1$$

Equation 2.

$$x^2+y^2=R^2$$

where

$$p = \sqrt{-R^4+2\,R^3x_3-R^2{x_3}^2+R^2-2\,Rx_3+{x_3}^2+{y_3}^2},$$
$$q = \sqrt{R^2-2\,Rx_3+{x_3}^2+{y_3}^2},$$
$$w = \sqrt{k^2-2\,km+2\,kn+m^2+2\,mn+n^2},$$
$$m = (2y_2\sqrt{-{x_1}^4+2{x_1}^3x_3-{x_1}^2{x_3}^2+(x_1-x_3)^2+{y_3}^2}$$
$$-2x_2{x_1}^2+2x_2x_1x_3+\sqrt{x_1-x_3^2+{y_3}^2}(x_1+x_3))^2,$$

$$n = (2x_2\sqrt{-{x_1}^4+2{x_1}^3x_3-{x_1}^2{x_3}^2+(x_1-x_3)^2+{y_3}^2}$$
$$+2y_2{x_1}^2-2y_2x_1x_3-y_3\sqrt{(x_1-x_3)^2+{y_3}^2})^2,$$
$$k = (2y_3\sqrt{-{x_1}^4+2{x_1}^3x_3-{x_1}^2{x_3}^2+(x_1-x_3)^2+{y_3}^2}$$
$$-2x_3{x_1}^2+2{x_3}^2x_1+\sqrt{x_1-x_3^2+{y_3}^2}(x_1+x_3))^2$$

and u and v are the rotated coordinates.

Solving the mentioned system we find the exact shape and Euler triple of the sought bicentric quadrilateral. In concrete cases this system can be solved using computer algebra.

Returning to the construction described in Theorem 1 we notice that the same method can be used to construct a bicentric n-gon for arbitrary $n > 4$. Starting with $n = 5$ one can also extend this approach to construct self-intersecting bicentric n-gons, the so-called bicentric n-stars (cf., e.g., [2]). In the last section of this paper, we present an explicit conjecture in the case of convex bicentric pentagons.

In the next section we discuss possible interpretation of our results in terms of the so-called Kendall shape space [5].

5. Bicentric shapes in Kendall shape space of quadrilaterals

The shape spaces introduced by D.Kendall [5] play important role in several applications of geometry (see, e.g., [5], [6]) so it seems interesting to interpret the above results in terms of Kendall shape space of quadrilaterals $S(4,2)$. It is well known that $S(4,2)$ is a four-dimensional compact smooth manifold without boundary which is diffeomorphic to the complex projective space CP^2 [5]. Recently, C.Klingenberg constructed natural coordinate systems in Kendall shape spaces [6].

In these terms, our Theorem 1 means that the pair of lengths of first two sides of bicentric quadrilateral give a natural coordinate system in the two-dimensional subset B_4 of $S(4,2)$ consisting of the shapes of bicentric quadrilaterals. In fact, comparison of our construction with the construction used by C.Klingenberg shows that our coordinate system is a restriction of the coordinate system constructed in [6]. So we are able to use the topological results of [6], which yields the following conclusion.

Corollary. The set of bicentric quadrilateral shapes B_4 is a two-dimensional surface in $S(4,2)$ diffeomorphic to the two-dimensional sphere S^2.

The described connection with the paper of C.Klingenberg suggested several further applications of our construction in the context of Kendall of shape spaces, which will be discussed in a forthcoming paper of the author.

6. Generalization to bicentric pentagons

In the sequel we briefly outline application of our approach to construction of bicentric pentagon. First of all, analyzing the proof of Theorem 1 one comes to the following conjecture which seems highly plausible.

Conjecture. For any three points A, B, C in a circumference S, there exist the fourth and fifth points D and E in S such that the pentagon $< A, B, C, D, E >$ is bicentric. Actually, there is good evidence that an analogous result should be true for bicentric n-gons but we do not discuss these aspects for the reason of space. Instead we present further details in the symmetric case, where $|AB| = |BC| = a$. In this case, using Theorem 1 and the method of proof of Theorem 2, we get the two equations with two variables y_t and v. So we can find y_t and v, where y_t is the coordinate of O_t and v is the distance from the center of S to the given fifth side.

In conclusion we present an example illustrating our main results. As was shown, in the symmetric case we have

$$r = a\left(\sqrt{4R^2 - a^2}\right) / \left(\sqrt{4R^2 - a^2} + a\right), \tag{6}$$

and using Fuss relations (1) and (6) we calculate d which has the following form

$$\begin{aligned} d = [\sqrt{2}((2R^2 + ak)(2kR^2a + 4R^4 + 4R^2a^2 - a^4 \\ -\sqrt{a^2k^2(16R^4 + 4R^2a^2 + 8R^2ak - a^4)}))^{1/2}]/[4R^2 + 2ak] \end{aligned} \tag{7}$$

where

$$k = \sqrt{R^2 - a^2}.$$

We now verify the above results for the concrete values $R = 5$, $A = (3, 4)$, $B = (0, 5)$. Here $|AB| = a = 3.16$ and equation (6) gives that $r = 2.37$ and $d = 2.5$, which can be independently confirmed using Geogebra.

REFERENCES

1. Radic M. Some relations concerning triangles and bicentric quadrilaterals in connection with Poncelet's closure theorem. *Math. Maced.*, **1** (2003), 35-58.

2. Radic M. Certain inequalities concerning bicentric quadrilaterals, hexagons and octagons. *Journal of Inequalities in Pure and Applied Mathematics*, **6**, 1, 1 (2005).

3. Bencze M., Dragan M. Some inequalities in bicentric quadrilateral. *Acta Univ. Sapientiae, Mathematica*, **5**, 1 (2013) 20-38.

4. Halbeisen L., Hungerbuhler N. A Simple Proof of Poncelet's Theorem, (on the occasion of its bicentennial). *Mathematical Assoc. of America, American Mathematical Monthly*, **121**, 1 (2014).

5. Kendall D. Shape manifolds, Procrustean metrics, and complex projective spaces. *Bulletin of the London Mathematical Society*, **16** (1984), 81-121.

6. Klingenberg C. Walking on Kendall's Shape Space: Understanding Shape Spaces and Their Coordinate Systems. *Evolutionary Biology*, **47** (2020), 334-352.

7. Josefsson M. When is a Tangential Quadrilateral a Kite?, *Forum Geometricorum*, **11** (2011) 165-174.

8. Radic M. Certain relations concerning bicentric polygons and tow-parametric representation of Fuss' relations. *Mathematica Pannonica*, **20**, 2 (2009), 219-248.

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