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INVERSE PROBLEM FOR SECOND ORDER REGULAR EQUATIONS AND LINE CONFIGURATION OF SINGULAR POINTS

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Abstract. In this paper we deal with the problem of construction of second order regular differential equations with polynomial coefficients on Riemann sphere with given monodromy, such problem arises from inverse problem of electrostatics and is related to controlability of quantum systems. We consider the particular case of this problem.

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1. Reduction of the Schrödinger equations to Fuchsian systems

It is known that certain physically interesting Schrödinger equations

$$i\frac{\partial\Psi(t)}{\partial t} = H(t)\Psi(t) \tag{1}$$

with time-dependent Hamiltonian $H(t) = (H_{ij}(t))$, $i, j = 1, ..., N$, where

$$H_{11} = \varepsilon(t), H_{12} = V_2, H_{13} = V_3, ..., H_{1N} = V_N,$$

$$H_{21} = V_2, H_{31} = V_3, ..., H_{N1} = V_N, \text{ and } H_{ij} = 0 \text{ otherwise},$$

$\Psi(t) = (\psi_1(t), ..., \psi_N(t))$ is a wave function, $V_j, j = 2, \dots, n$ are constants, and the time-dependent part ε has the form $\varepsilon(t) = E_1 \tanh(t/T)$ with constant E_1 and T can be rewritten as Fuchsian systems [4] of special type

$$(z\mathbb{I} - B)\dot{X} = AX, \tag{2}$$

where X is a complex-valued (n-column) vector function of complex variable z, dot denotes differentiation over z and

$$B = s_1\mathbb{I}_{n_1} \oplus ... \oplus s_p\mathbb{I}_{n_p},\ s_i \in \mathbb{C}; s_i \neq s_j,\ \text{when } i \neq j; n_1 + ... + n_p = n$$

and $A \in End(n, \mathbb{C})$.

In particular equation (1) is reducible to an N-dimensional Fuchsian system of type (2)

$$(zI_N - B)\frac{d\Phi(z)}{dz} = A\Phi(z) \tag{3}$$

with $B = \operatorname{diag}(\mathrm{i}, -\mathrm{i}, \ldots, -\mathrm{i})$.

In this way one can construct Schrödinger equations with prescribed qualitative properties of solutions through monodromy representation of Fuchsian system (see, e.g., [4]). Namely, for $N = 2$, we obtain a Schrödinger equation with the two-component phase function

$$i\frac{\partial f(t)}{\partial t} = H(t)f(t), \tag{4}$$

where $f(t) = (f_1(t), f_2(t))$, and time-dependent Hamiltonian $H(t)$ has the form

$$H(t) = \begin{pmatrix} \varepsilon(t) & V(t) \\ V(t) & -\varepsilon(t) \end{pmatrix} \tag{5}$$

where

$$\varepsilon(t) = \frac{E_0T + E_1Ty}{1+y^2}\frac{dy}{dt}, \quad V(t) = \frac{V_0T}{\sqrt{1+y^2}}\frac{dy}{dt}$$

with some monotonically increasing differentiable function $y : \mathbb{R} \to \mathbb{R}$ satisfying $y(t) \to \pm\infty$ as $t \to \pm\infty$.

In this case equation (4) is reducible to a system of two hypergeometric equations of the form

$$z(z-1)\frac{d^2g}{dz^2} + (\gamma - (1+\alpha+\beta)z)\frac{dg}{dz} - \alpha\beta g(z) = 0 \tag{6}$$

with appropriate constants α, β, γ.

The equation (6) is the two-dimensional case of (2) in which

$$A = \begin{pmatrix} 1-\gamma & 1 \\ (\alpha-\gamma+1)(\gamma-\beta-1) & \gamma-\alpha-\beta-1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is known that, for every hypergeometric equation (6), there exists a Fuchsian system with the same singular points and the same monodromy. This system has the form:

$$\frac{dF(z)}{dz} = \begin{pmatrix} 0 & 0 \\ -\alpha\beta & \gamma \end{pmatrix}\frac{F(z)}{z} + \begin{pmatrix} 0 & 1 \\ 0 & \gamma-(\alpha+\beta) \end{pmatrix}\frac{F(z)}{z-1}, \tag{7}$$

where $F(z)$ is a 2-dimensional vector function. Equation (6) and the system (7) gives two parameter monodromy representation

$$\rho_{\alpha,\beta,\gamma} : \pi_1(\mathbb{CP}^1 \setminus \{0,1,\infty\}) \to GL(2,\mathbb{C}). \tag{8}$$

Inverse problem, i.e, construct the system of the type (7) by monodromy representation (8) provides the desired properties of the solutions of (4).

In a similar way (1) type equation can be reduced to the Heun equation

$$\frac{d^2u}{dt^2} + \left(\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-d}\right)\frac{du}{dt} + \frac{\alpha\beta t - q}{t(t-1)(t-d)}u = 0. \tag{9}$$

Here $d \in \mathbb{C}$, is a parameter $(d \neq 0, 1)$, and $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$ are exponent-related parameters. The Riemann P-symbol is

$$P \left\{ \begin{array}{ccccc} 0 & 1 & d & \infty & \\ 0 & 0 & 0 & \alpha & ; t \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{array} \right\}.$$

This does not uniquely specify the equation and its solutions, since it omits the accessory parameter $q \in \mathbb{C}$. The exponents are constrained by

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0. \tag{10}$$

This is a special case of Fuchs's relation, according to which the sum of the $2n$ characteristic exponents of any second-order Fuchsian equation on $\mathbb{CP}^1$ with n singular points must equal $n - 2$.

The canonical form of Fuchsian system of the form (9) with exponents $(0, 1/2), (0, 1/2), (0, 1/2)$ and $(l/2, -(l+1)/2)$ is the Lamé equation

$$y^{''}(x) + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-s}\right) y^{'}(x) - \frac{l(l+1)x + 4q}{4x(x-1)(x-s)} y(x) = 0. \tag{11}$$

If $\alpha = 1$, $q = 1$, or $\alpha = 0$, $q = 0$ then the equation (9) reduces to the hypergeometric equation.

2. Linear configuration of singular points of the equation

The motion of charged particles using time varying electric fields is described by the Mathieu equation [10]:

$$w^{''}(\tau) + (a - 2q\cos(2\tau))w(\tau) = 0.$$

By the change of variable $z = sin^2\tau$ or $z = cos^2\tau$ we obtain the algebraic form of the Mathieu equation:

$$z(1-z)w^{''} + \frac{1}{2}(1-2z)w^{'} + \frac{1}{4}(a - 2q(1-2z))w = 0 \tag{12}$$

or

$$(1-z^2)w^{''} - zw^{'} + (a + 2q - 4qz^2)w = 0, \tag{13}$$

respectively.

The equations (12),(13) are the confluence Heun's equations and have regular singularity at 0, 1 of equation (12) and at -1, 1 of equation (13) both with exponents 0, $\frac{1}{2}$ and irregular singular point at ∞.

The *direct problem* of electrostatics for the Coulomb potential

$$E_Q = \sum_{i \neq j} \frac{q_i q_j}{d_{ij}}, \tag{14}$$

where d_{ij} is the distance between the points p_i and p_j (see [6], [9]) is the following problem: *given a compact conductor X and collection of m positive* *numbers $Q = (q_i)$, find all equilibrium configurations of charges q_i in X and determine their types as critical points of the Coulomb potential $E_Q|X_m$, where X_m is the space of m different points in X.*

Accordingly the inverse problem is the following [9]: *given a finite configuration $P = (x_1, ..., x_m)$ of points in a fixed subset X, does there exist a collection of nonzero real numbers $Q = (q_1, ..., q_m)$, interpreted as values of point charges, such that if they are placed at the points x_j then the given configuration is a critical point of Coulomb energy E_Q restricted to X_m?*

In [5] analyzed is the the inverse problem for the Coulomb potential. Namely, the following model considered. Let N be the number of point charges. Then the resultant force on q_i in position x_i corresponding to potential (14) is given by

$$F_m = \frac{q_m t_1}{x_m^2} + \sum_{j=1}^{m-1} \frac{q_m q_j}{(x_m - x_j)^2} - \sum_{j=m+1}^{N} \frac{q_m q_j}{(x_j - x_m)^2} - \frac{q_m t_2}{(L - x_m)^2}, \quad m = 1, ..., N.$$

The relations

$$F_1 = 0, F_2 = 0, ..., F_N = 0,$$

give a system of non-homogeneous linear equations

$$MQ = G, \tag{15}$$

for unknowns $q_1, ..., q_N$, where

$$G = \left(\frac{t_2}{(L - x_1)^2} - \frac{t_1}{x_1^2}, \frac{t_2}{(L - x_2)^2} - \frac{t_1}{x_2^2}, ..., \frac{t_2}{(L - x_N)^2} - \frac{t_1}{x_N^2} \right)^T,$$

$M = (m_{ij})_{i,j=1}^N$ is an antisymmetric matrix and $m_{ij} = (-1)^{\tau_{ij}} (x_j - x_i)^{-2}$, $i \neq j$, $\tau_{ij} = 0$, if $i > j$; and $\tau_{ij} = 1$, if $i < j$. Therefore, the solution of the system15 depends on the rank of the matrix M and from this inverse problem reduces investigation of the system of polynomial equation $detM$=0.

Similarly can be considered the inverse problem for other type potentials (Riesz, logarithmic and e.t.). Below we discuss some aspects of the inverse problem for the logarithmic potential

$$E(x_1, ...x_n) = -q_0 \sum_{j=0}^{n} \lg |p_0 - x_j| - ... - q_m \sum_{j=0}^{n} \lg |p_m - x_j - \\ - \sum_{1 \le i < j \le n} \lg |x_j - x_i| \tag{16}$$

and reduce this problem to the inverse problem from the analytic theory of differential equations, in particular, on the monodromy problem for regular equations.

Problem. (see [6]) For given pairs $(p_0, q_0), ...(p_m, q_m)$, where $p_0 < ... < p_m$, $p_j \in \mathbb{R}$, $q_j > 0$, $j = 1, ..., m$ and given monodromy representation

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{p_0, ..., p_m\}, x_0) \to GL(2, \mathbb{C}) \tag{17}$$

construct a generalized Lamé equation

$$y^{''}(x) + \sum_{j=0}^{m} \frac{q_j}{x - p_j} y^{'}(x) + \frac{C(x)}{\prod_{j=0}^{m}(x - p_j)} y(x) = 0, \qquad (18)$$

where $C(x)$ is a polynomial of degree $m - 2$, such that the monodromy representation of (17) coincides (18).

This is Riemann-Hilbert monodrmy problem (Hilbert's 21 problem) for regular equations with some restriction of coefficients. The solution of the this problem in this form has, necessarily, apparent singularities besides the given singularities S (see [2], [7]).

It is known that the number of parameters determining a Fuchsian equation of order p with n singular points is less than the dimension of the space of monodromy representations, if $p > 2$, $n > 2$ or $p = 2, n > 3$. Hence in the construction of a Fuchsian equation with the given monodromy there arise the so-called apparent singularities at which the coefficients of the equation have poles but the solutions are single-valued meromorphic functions, i. e., the monodromy matrices at these points are identity matrices.

Remark. In the general case, for the differential equation of n order with m singular points (i.e. $cardS = m$, where S is a set of singular poits of equation), the totality of representations of $\pi_1(M - S)$ to $GL(n, C)$ and the totality of the corresponding Fuchsian differential equations form complex manifolds of dimension $n^2(m + 2g - 2) + 1$, and $(n^2(m + 2g - 2)/2) + (nm/2)$, respectively, where M is a compact Riemann surface of genus g. The difference of these dimensions is equal to the number N of apparent singular points

$$N = 1 - n(1 - g) + \frac{n(n-1)}{2}(m + 2g - 2), \quad (m = \#S)$$

when the monodromy representation is irreducible [2].

We consider the particular case of the above mentioned problem.

Proposition 1. *Let given pairs* $(-1, q_0), (1, q_1)$ *and irreducible monodromy representation*

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{-1, 1, \infty\}) \to GL(2, \mathbb{C}).$$

Suppose $\alpha = 2q_1 - 1$ *and* $\beta = 2q_0 - 1$ *are positive numbers. Then the monodromy representation of the differential equation*

$$(1 - x^2)y^{''} + (\beta - \alpha - (\alpha + \beta + 2)x)y^{'} + \gamma y = 0, \qquad (19)$$

where γ *is a constant, coincides with* ρ.

Indeed, Riemann-Hilbert monodrmy problem is solvable for the second order Fuchsian system without accessory parameters if and only if the number of singular points is equal to 3. The irreducibily properties of the representation ρ garanties existence of the numbers $\alpha > -1$, $\beta > -1$ such that $\alpha = 2q_1 - 1$ and $\beta = 2q_0 - 1$.

Proposition 2. *If $\gamma = n(n+\alpha+\beta+1)$ for some natural $n = 1, 2, 3, ...$, then equation (19) has polynomial solution $P_n^{\alpha,\beta}(x)$ of degree n.*

Polynomial functions $P_n^{\alpha,\beta}(x)$ from Proposition 2 are Jacobi polynomials of degree n and the proof of proposition directly follows from main properties of Jacobi polynomials [12].

$P_n^{(\alpha,\beta)}(x)$ can be the given explicitly by the expression

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2}\sum_{j=0}^{n} \frac{(n+\alpha)!}{(n-j)!}\frac{(n+\beta)!}{j!}(x-1)^j(x+1)^{n-j}.$$

From the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}(x-1)^{-\alpha}(x+1)^{-\beta}\left(\frac{d}{dx}\right)^n [(x-1)^{n+\alpha}(x+1)^{n+\beta}]$$

it follows, that $P_n^{(\alpha,\beta)}(x)$ are analytic functions of the parameters $\alpha, \beta \in \mathbb{C}$. The roots $x_1, ..., x_n$ of the equation $P_n^{(\alpha,\beta)}(x) = 0$ are real, satisfy inequalities $-1 < x_1 < x_2 < ... < x_n < 1$ and are analytic as functions α, β. Besides, $\frac{\partial x_j}{\partial \alpha} > 0$ and $\frac{\partial x_j}{\partial \beta} < 0$, $j = 1, ..., n$ [1].

In this situation the potential (16) takes the form

$$E(x_1, ...x_n) = -q_0\sum_{j=0}^{n} \lg|1-x_j| - q_1\sum_{j=0}^{n} \lg|1-x_j| - \sum_{1\le i<j\le n} \lg|x_j - x_i|. \quad (20)$$

The stationary points of E satisfy the system of equations

$$\frac{q_0}{1+x_j} - \frac{q_1}{1-x_j} - \sum_{1\le i<j\le n} \frac{1}{x_j - x_i} = 0, \quad i, j = 1, ..., n, \quad i \neq j$$

and coincide with roots of Jacobi polynomials.

The above mentioned mathematical formalism is description of the electrostatic model of m unit point charges on line (see [1], [12]).

Let two positive fixed charges of mass $\frac{\beta+1}{2}$ and $\frac{\alpha+1}{2}$ at -1 and $+1$, respectively and allow m positive unit charges $X = \{x_1, ..., x_m\}$ to move freely in $(-1, 1)$. The total energy $E(X)$ of this system if the interaction obeys the logarhitmic potential law is equal to

$$E(X) = E_{int} + E_{ext},$$

where

$$E_{int} = -\sum_{1\le k\le j\le n} \ln|x_k - x_j|,$$

and

$$E_{ext} = \sum_{k=1}^{n} \varphi(x_k)$$

with the external field $\varphi(x)$ created by the fixed charges:

$$\varphi(x) = -\frac{\beta+1}{2}\ln|x+1| - \frac{\alpha+1}{2}\ln|x-1|. \tag{21}$$

Then there exists a unique configuration $X^* = \{x_1^*, ..., x_m^*\}$ providing the global minimum of $E(X)$ in $[-1,1]^m$, corresponding to the unique equilibrium position for given free charges, and the points x_j^* are the zeros of the polynomial $P_n^{(\alpha,\beta)}$.

The critical points of the energy functional $E(X)$ as the function of x_j are the solutions of the equation

$$\frac{\partial}{\partial x_k}E(X) = 0.$$

Suppose X^* is a critical configuration, then

$$\frac{\partial}{\partial x_k}E_{int}(X)|_{X=X^*} + \varphi^{'}(x_k) = 0 \tag{22}$$

Suppose $y(x) = (x - x_1^*)(x - x_2^*)...(x - x_n^*)$ is a monic polynomial with zeros at x_k^*'s, then

$$\frac{\partial}{\partial x_k}E_{int}(X)|_{X=X^*} = -\sum_{1\le j\le n, j\neq k}\frac{1}{x_k^* - x_j^*} = -\frac{1}{2}\frac{y^{''}(x_k^*)}{y^{'}(x_k^*)}$$

and

$$\varphi^{'}(x) = -\frac{\beta+1}{2(x+1)} - \frac{\alpha+1}{2(x-1)}.$$

From (22) we obtain

$$y^{''}(x) + \left(\frac{\beta+1}{x+1} + \frac{\alpha+1}{x-1}\right)y^{'} = 0 \tag{23}$$

for all $x \in X^*$.

From equation (23) we obtain that the polynomial

$$(1-x^2)y^{''}(x) + (x(\alpha+\beta+2) + (\alpha-\beta))y^{'}$$

of degree n is equal to zero at the zeros of polynomial $y(x)$ and therefore equal to $\texttt{const} \times y(x)$. Denote by γ this constant, we obtain a second order differential equation (19) (see [12]).

Finally we remark, that if we return to equation (4) we obtain the Shrödinger equation depending on parameters [11]. We consider these parameters as control parameters. The problem of this type arises for the generate universal set of quantum gates for quantum processor [3-4], [11] or to obtain necessary configuration of ion in trap ions technology [8].

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