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On Transporting Hydrogen via Pipelines Through Georgia Amid Climate Change Challenges

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This study explores the unsteady flow dynamics of a high-pressure hydrogen and natural gas mixture within a gas pipeline. The fluid is modeled as a homogeneous blend of hydrogen and natural gas, with an averaged density derived from the individual densities of both gases under the allowance of a polytropic process. The investigation focuses on three principal dependent variables: fluid pressure, temperature, and flow rate, which are influenced by a set of nonlinear hyperbolic partial differential equations. The research addresses a quasi-nonlinear one-dimensional partial differential equation system, characterizing the transient flow behaviors of the combined natural gas and hydrogen substances through a pipeline. The paper offers analytical solutions to these one-dimensional partial differential equations, delineating the flow of isothermal gas through pipelines that are both inclined and have inclined branches.

Keywords: Hydrogen-natural gas mixture; Inclined and branched pipelines; Partial differential equations; Analytical solutions

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1. Introduction

In the last four decades, accelerated changes in global and regional climates, primarily due to anthropogenic activities, have been well-documented (IPCC, 2018). These changes manifest as a warming atmosphere and oceans, rising sea levels, rapid glacier melting, and significant shifts in precipitation patterns. The proximity of climate change to a critical red line, beyond which limiting global warming to safe levels becomes exceedingly challenging, is now evident (IPCC, 2018). The magnitude of future climate shifts will be directly related to our success in curtailing anthropogenic greenhouse gas emissions: the more we emit heat-trapping gases and soot today, the more severe the impact on our climate (Keshavjee, 2009). In alignment with the Paris Agreement's objectives to curb climate change, achieving net-zero global carbon emissions by 2050 is imperative, demanding significant reductions in fossil fuel emissions and a swift transition towards low-carbon and renewable energy sources (IPCC, 2018).

Hydrogen stands out as a particularly promising option, with the potential to largely replace natural gas for residential and industrial applications. Notably, the combustion of hydrogen yields water, not the greenhouse gas carbon dioxide (CO2),

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thus preventing contributions to climate change (Ball et al., 2015). Despite being the simplest and most abundant element in the universe, hydrogen does not naturally occur in its free form on Earth and must be extracted from compounds containing hydrogen atoms (Ball et al., 2015). The majority of hydrogen is derived from fossil fuels, especially natural gas via steam reforming of methane, and from water electrolysis using electricity from renewable sources like biomass, geothermal, hydropower, wind, and solar energy (Ball et al., 2015; Hosseini and Wahid, 2016). Consequently, hydrogen fuel offers a compelling alternative to fossil fuels in mitigating recent climate change effects due to its potential to substantially lower CO2 emissions when replacing conventional energy sources (Elaoud and Hadj-Taieb, 2008). Presently, economic strategies worldwide are focused on diminishing fossil fuel usage in power generation, elevating the role of hydrogen as a future energy carrier (Shen et al., 2017). The versatility of hydrogen is expected to extend beyond electricity generation to applications such as fuel production and storage for automobiles and aviation, as well as for diverse industrial and domestic energy requirements (Elaoud and Hadj-Taieb, 2008). However, the large-scale production of hydrogen necessitates extensive storage solutions, including hydrogen storage in porous rocks, subterranean salt caverns, and depleted oil and gas fields (Pfeifere et al., 2017; Heinemann et al. 2018). Additionally, the efficient transport of hydrogen gas represents a central challenge to many researchers worldwide (Heinemann et al. 2018; Ball et al. 2015; Hosseini and Wahid, 2016; Shen et al., 2017). Given the prohibitive costs of establishing new hydrogen transport infrastructure, repurposing existing gas pipeline networks by transporting a mixture of hydrogen and natural gas appears economically viable. Burst tests have indicated that hydrogen blended with natural gas does not compromise the mechanical integrity of API X52 steel, the most commonly used material in current pipelines (Capelle et al., 2008; Tabkhi et al., 2008). Thus, the investigation of the flow dynamics of hydrogen and natural gas mixtures in pipelines is a pressing research need that has garnered the attention of many scientists (Shen et al., 2017; Marangon and Carcassi, 2014; Stolecka, 2018). Introducing hydrogen into the natural gas supply involves an array of technical and safety considerations. Testing has shown that up to 15% hydrogen volume does not significantly alter pipeline network reliability, yet concentrations between 15-50% or higher dramatically affect the structural integrity of steel. Each network must individually assess the maximum permissible hydrogen concentration while keeping pipeline safety in mind. For example, networks in the Netherlands and Germany limit hydrogen levels to under 6% (Rusin and Stoletska, 2011).

Georgia's national energy agenda now includes hydrogen as a clean fuel option in response to regional climate change impacts, such as the increased frequency of extreme weather events within the Caucasus region, and particularly Georgia (Georgia's Third National Communication to the UNFCCC, 2015). The high efficiency and low emission profile of hydrogen offers Georgia an opportunity to enhance electric power generation. Additionally, there are times when electricity produced from various renewable sources (hydroelectric, wind, and solar) leads to excess energy supply during off-peak hours, which could be utilized for green (electrolysis-based) or blue (petroleum-derived) hydrogen production (Georgia Energy Profile, 2023). Furthermore, Georgia is strategically positioned to develop transit networks between the Black Sea and Central Asia, thus optimizing its geopolitical leverage.

In the light of the recent conflict between Russia and Ukraine, European Union (EU) countries are actively seeking alternative routes for the transportation of gas and oil from Eastern regions, with Kazakhstan, Turkmenistan, and Azerbaijan emerging as potential conduits. Both Kazakhstan and Turkmenistan, in particular, are poised to become leaders in "green" and "blue" hydrogen production, given their vast resource potential and advanced petrochemical infrastructures (UNECE, 2023). Plans are underway to construct the Trans-Caspian Pipeline (TCP), designed to interface with Turkmenistan's internal East-West land pipeline and connect to the Azerbaijani Sangachal terminal. From there, the pipeline will integrate with the South Caucasus Pipeline (SCP), extending through Georgia and Turkey and heading towards EU domains. This new arterial network is a response to the EU's goals for a hydrogen-driven energy transition, positioning non-Russian natural gas (and blue hydrogen) sources as a more attainable and cost-effective alternative for the shift (South Caucasus Pipeline, 2022). Moreover, with China commanding the global hydrogen market by producing 20 million tons annually, which accounts for about a third of the worldwide output, the EU serves as a prime market for these products (Bocca et al., 2023).

The primary objective of this research is to forge a theoretical (analytical) framework for forthcoming investigations into the transportation of "green" and "blue" hydrogen-natural gas mixtures across inclined and branched pipeline systems, such as the proposed route of the South Caucasus gas pipeline. Consequently, this study delves into the theoretical (analytical) examination of mixture transport dynamics within inclined and branched pipeline architectures.

2. Density equation for a mixture of hydrogen and natural gas

For the polytropic flow in a pipeline, the mass ratio and density of a hydrogen-natural gas mixture can be described as follows (Elaoud, Hadj-Taieb, 2008; Davitashvili, Rukhaia, 2023). Assume that the flow is one-dimensional and pertains to a homogeneous mixture of hydrogen and natural gas. The computation of pressure loss in the pipeline is analogous to that in steady-state flows. The mass ratio of hydrogen to natural gas is given by equation (2.1):

$$\eta = \frac{m_h}{m_h + m_g} \tag{2.1}$$

where m_h and m_g denote the masses of hydrogen and natural gas, respectively.

The mixture's density is defined by equation (2.2):

$$\rho = \frac{m_h + m_g}{V_h + V_g} \tag{2.2}$$

where $V_h = \frac{m_h}{\rho_h}$ and $V_g = \frac{m_g}{\rho_g}$ are the volumes of hydrogen and natural gas respectively, influenced by their densities ρ_h and ρ_g.

By incorporating equation (2.1) into the definition of density, we derive equation (2.3):

$$\rho = \left[\frac{\eta}{\rho_h} + \frac{1-\eta}{\rho_g}\right]^{-1} \tag{2.3}$$

Note that under standard conditions (0C, 1.01325 bar), the molar mass (density) of hydrogen ρ_h is approximately eight times less than that of natural gas ρ_g (methane). Furthermore, the specific heat capacity of H_2 is roughly 6.5 times greater than that of CH_4 (Klopci‘c et al. 2022). As such, the density of a hydrogen and natural gas mixture can vary according to the laws of polytropic and adiabatic flow processes, among others. The density changes according to the polytropic relationship given by equation (2.4):

$$\rho_h = \rho_{h0}\left(\frac{p}{p_0}\right)^{\frac{1}{n_1}}, \quad \rho_g = \rho_{g0}\left(\frac{p}{p_0}\right)^{\frac{1}{n_2}} \tag{2.4}$$

where ρ_{h0} is the initial density of hydrogen, ρ_{g0} the initial density of natural gas, p_0 the initial pressure, and n_1 and n_2 their respective polytropic exponents. These exponents are related to the specific heats for each gas and are crucial for characterizing the flow of the gas mixture.

Upon substituting (2.4) into (2.3), we obtain the density equation for the mixture considering both hydrogen and natural gas under polytropic processes as given by the following expression (Elaoud and Hadj-Taieb, 2008; Chaharborj and Amin, 2020):

$$\rho = \left[\frac{\eta}{\rho_{h0}}\left(\frac{p_0}{p}\right)^{\frac{1}{n_1}} + \frac{1-\eta}{\rho_{g0}}\left(\frac{p_0}{p}\right)^{\frac{1}{n_2}}\right]^{-1} \tag{2.5}$$

The subscript ‘zero‘ denotes initial conditions. The equation of state allows for the determination of initial density based on given initial pressure and temperature. Using the equation of state, we calculate the initial pressure as follows:

$$p_0 = \rho_{g0}\frac{ZR_gT_0}{M_1} = \rho_{g0}\cdot Z_g \cdot R_g \cdot T_0 \tag{2.6}$$

Alternatively, for the hydrogen:

$$p_0 = \rho_{h0}\frac{ZR_hT_0}{M_2} = \rho_{h0}\cdot Z_h \cdot R_h \cdot T_0 \tag{2.7}$$

Here, M_i for $i = 1, 2$ represents the molecular masses of hydrogen and natural gas, respectively, Z is the compressibility factor, and R_h and R_g are the specific gas constants for hydrogen and natural gas. The constant $R_h = \frac{R_u}{M_h}$ and $R_g = \frac{R_u}{M_g}$, where R_u is the universal gas constant.

Equation (2.5) illustrates that the mixture's density variability in polytropic flow within a pipeline depends on the initial mass values of both gases and the changing pressure along the pipeline. This study aims to ascertain the pressure and flow rate of gas in an inclined and branched pipeline system, given a specified mass ratio of hydrogen to natural gas.

The European Union (EU) regards the Transport Corridor Europe-Caucasus-Asia (TRACECA) as essential for fostering cooperation and ensuring peace and stability in the Caucasus region. The Baku-Tbilisi-Ceyhan (BTC) oil pipeline and the South Caucasus Gas Pipeline (Baku-Tbilisi-Erzurum gas pipeline, SCP, or BTE) are critical infrastructures that transport Caspian energy resources from Azerbaijan through Georgia to Turkey and then to European markets. The BTE pipeline spans 692 km, with a 1070 mm diameter, and has an annual capacity to transport 24 billion m^3 of gas. A 248 km segment crosses Georgian territory, featuring two pressure reduction stations, two compressor stations, and one intermediate station (SCP, 2022). With plans to connect to Turkmen and Kazakh gas fields via the proposed Trans-Caspian Gas Pipeline, Azerbaijan is considering the pipeline's expansion, intending to increase the capacity up to 60 billion m^3 annually by constructing a second parallel pipeline (Sokor, 2014). Additionally, the Baku-Supsa oil pipeline (Western Export Pipeline Route), Mozdok-Tbilisi-Yerevan (Armenia) gas pipeline, and the Chmi (Russia)-Tbilisi-Saguramo gas pipelines are operational in Georgia (SCP, 2022) and exhibit complex design features such as pressure reduction stations, compressor stations, branches, slopes, and traverse heterogeneous terrain. Consequently, the study of gas and liquid flow dynamics in inclined and branched pipelines has emerged as a critical area of research in the Caucasus region, with Georgia being a geographical focal point.

3. Gas flow in the inclined pipeline

In an inclined gas pipeline, gas flow is governed by the fundamental dynamical conservation laws of mass, momentum, and energy. For one-dimensional gas flow dynamics through such a pipeline, this behavior is described by the following set of partial differential equations (PDEs) (Chaczykowski, 2010; Herrn-Gonzlez et al., 2009):

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{3.1}$$

$$\frac{\partial (\rho v S)}{\partial t} + \frac{\partial \left(PS + \rho v^2 S\right)}{\partial x} + |\tau| \pi D + \rho S g \sin\theta = 0, \tag{3.2}$$

$$\frac{\partial}{\partial t}\left[\left(e + \frac{v^2}{2}\right)\rho S\right] + \frac{\partial}{\partial x}\left[\left(h + \frac{v^2}{2}\right)\rho v S\right] - \Omega + \rho S g v \sin\theta = 0, \tag{3.3}$$

where $\rho(x,t)$ represents the gas density, $p(x,t)$ the pressure, $v(x,t)$ the velocity, T the temperature, S the cross-sectional area of the gas duct, D the duct diameter, θ the pipe inclination angle, e the specific internal energy, and h the specific enthalpy, defined as $h = e + \frac{p}{\rho}$. Moreover, τ denotes the tangential stress between the gas and the inner wall of the pipeline, while Ω represents the heat transfer between the gas and its environment per unit length (the thermal flow term).

As the system of equations (3.1)(3.3) encompasses three equations with four unknown variables, it is essential to introduce an additional equation to represent the state of the gas:

$$p = \rho \frac{Z R_4 T}{M} = \rho \cdot Z \cdot R_g \cdot T, \tag{3.4}$$

where M signifies the molecular mass of the gas mixture, Z is the variable compressibility factor, $R_g = \frac{R_u}{M}$ symbolizes the gas constant, and R_u is the universal gas constant.

To resolve the nonlinear equation system (3.1) - (3.4), the heat flow term Ω must be determined. We consider three significant scenarios:

(a) Isothermal flow (T = constant), wherein the temperature inside the gas does not change significantly and the heat exchange with the surroundings can be discounted. In this instance, the energy equation is superfluous.
(b) Adiabatic flow ($\Omega = 0$), corresponding to rapid gas dynamics where slow thermal conduction is negligible (this includes the special case of isentropic flow, which maintains constant entropy).
(c) Non-thermal equilibrium, where the flue and the ground are not thermally balanced, Ω is non-zero, necessitating an additional equation to model heat transfer.

In scenario (c), we consider a compressible fluid flowing within a pipe with a constant wall temperature T_w (usually, the temperature of the gas T exceeds the wall temperature Tw). Assuming heat exchange is governed only by turbulence and thermal conduction along the pipeline wall, the following equation is used (Chaczykowski, 2010):

$$\Omega = \frac{-K_L (T - T_w)}{C_p M} + \frac{\partial}{\partial x} \left(\lambda \frac{\partial v}{\partial T} \right) \tag{3.5}$$

where K_L is the heat transfer coefficient, M is the mass flow rate, and λ is the thermal conductivity coefficient. The systems internal energy, denoted as e, is composed of molecular kinetic energy, potential energy from vibrational motion, and the atomic electrical energy within the molecules. Evidently, the internal energy e is a function of both the temperature T and the number of gas moles n. Enthalpy h, on the other hand, is the summation of the internal energy e and the product of pressure P and volume V of the system. Therefore, the alterations in internal energy and enthalpy are primarily tied to temperature changes within the system.

Taking into account the general relation for the change of enthalpy $h = e + \frac{p}{\rho}$ for a pure substance and the dependence of enthalpy on temperature and pressure $h = h(T, P)$, we have:

$$\mathrm{d}h = \left(\frac{\partial h}{\partial T} \right)_P \mathrm{d}T + \left(\frac{\partial h}{\partial P} \right)_T \mathrm{d}P \tag{3.6}$$

where $C_p = \left(\frac{\partial h}{\partial T}\right)_P$ is the heat capacity at constant pressure, and

$$\left(\frac{\partial h}{\partial P}\right)_T = \left[v - T\left(\frac{\partial v}{\partial T}\right)_P\right] \mathrm{d}P$$

represents the Joule-Thomson coefficient. Consequently, we derive the following expression for the differential of enthalpy:

$$\mathrm{d}h = C_p \mathrm{d}T + \left[v - T\left(\frac{\partial v}{\partial T}\right)_P\right] \mathrm{d}P \tag{3.7}$$

Inserting Eq. (3.5) into Eq. (3.3) and considering Eq. (3.6), we obtain the updated form of Eq. (3.3) as provided by Chaczykowski (2010):

$$\frac{\partial T}{\partial t} + \frac{v}{C_v}\frac{\partial T}{\partial x} - \frac{\partial P}{\partial t} + \rho S g v \sin\theta = \frac{-K_L(T - T_w)}{C_p M} + A\frac{\partial}{\partial x}\left(\lambda \frac{\partial v}{\partial T}\right) \tag{3.8}$$

From a practical standpoint, expressing consumer demand in terms of mass flow, it is logical to substitute the velocity v in equations (3.1)-(3.3) with the flow rate q, as defined by:

$$q = \rho v S = \rho Q \tag{3.9}$$

Friction can also be characterized through the Fanning friction factor f using the following relationship:

$$f = \frac{2|\tau|}{\rho v^2} \tag{3.9}$$

Acknowledging Eqs. (3.8) and (3.9) in conjunction with Eqs. (3.1) and (3.2), we derive the revised equations:

$$\frac{\partial \rho}{\partial t} + \frac{1}{S}\frac{\partial q}{\partial x} = 0, \tag{3.10}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(Sp + \frac{q^2}{S\rho}\right) + \frac{2fq|q|}{DS\rho} + \rho S g \sin\theta = 0. \tag{3.11}$$

Using the relation $P = c^2\rho$, which is accurate within transition gas pipelines, and taking into account Eq. (3.8), we can derive:

$$\frac{\partial P}{\partial t} + \frac{c^2}{S}\frac{\partial q}{\partial x} = 0 \tag{3.12}$$

$$\frac{\partial q}{\partial t} + S\frac{\partial P}{\partial x} + \frac{c^2}{S}\frac{\partial}{\partial x}\left(\frac{q^2}{P}\right) + \frac{2fc^2 q|q|}{DSP} + \frac{Sg\sin(\theta)}{c^2}P = 0 \tag{3.13}$$

The system of equations (3.12), (3.13), (3.7), and (3.4), with additional constraints $0 \leq x < L$, $t \geq 0$, is solved with the following initial conditions:

$$P(x,0) = P_0(x), \quad q(x,0) = q_0(x), \quad T(x,0) = T_0(x), \tag{3.14}$$

and boundary conditions:

$$P(0,t) = P_1(t), \quad q(L,t) = q_1(t), \quad T(0,t) = T_1(t), \quad \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0. \tag{3.15}$$

The system composed of equations (3.12), (3.13), (3.7), and (3.4), with initial conditions (3.14) and boundary conditions (3.15), was solved by the Lax-Wendroff finite difference method. The initial values $P_0(x)$, $q_0(x)$, and $T_0(x)$ were determined based on the steady-state equations for gas motion (Davitashvili, 2021):

$$P_0(x) = \sqrt{P_1^2 - \left(P_1^2 - P_2^2\right)\frac{x}{L}},$$

$$T_0(x) = T_w + \left(T_0(0) - T_w\right)e^{-\beta x},$$

$$q_0(x) = Q_{\text{std}} \cdot \left(\frac{T_0(x)}{T_{\text{std}}}\right) \cdot \left(\frac{P_{\text{std}}}{P_0(x)}\right),$$

where P_1 and P_2 are the pressures at the inlet and outlet of the pipeline, respectively, L is the length of the pipeline, Q_{std} is the standard flow rate of 84.78 L/min, T_{std} is the standard temperature of 21.11C, and P_{std} is the standard absolute pressure of 101.3 kPa. The value of β is derived from the boundary conditions (Davitashvili, 2021):

$$\beta = -\frac{1}{L}\ln\left(\frac{T(L,0) - T_w}{T(0,0) - T_w}\right).$$

For the setup considered, the flow of a compressible fluid in a pipe with a constant wall temperature T_w is analyzed, taking into account that the heat exchange occurs solely due to turbulence and thermal conductivity along the pipeline wall. Several experiments were conducted with varying mass ratios of hydrogen to natural gas, as well as different pressure values, turbulence intensities, and wall temperatures.

Unfortunately, an approximate analytical solution to the system of equations (3.12), (3.13), (3.7), (3.4) with initial and boundary conditions (3.14)-(3.15) is not yet known. Therefore, to obtain any form of analytical solution to the problem, certain simplifications are necessary. Assume that there is an isothermal flow, where the temperature change within the gas is negligible such that thermal conduction between the gas and environment can be discounted. It has been assumed that the gas flow within the pipeline is isothermal and that pressure waves propagate through the gas at the speed of sound without attenuation. Considering a high-pressure gas pipeline where dynamic changes span several hours, we can estimate the magnitude of each term in the system of equations (3.2)-(3.3) using typical values for the involved variables as calculated in (Herran-Gonzalez, 2009). This allows us to disregard the third term and the first term in the momentum equation:

$$\frac{\partial P}{\partial t} + \frac{c^2}{S}\frac{\partial q}{\partial x} = 0, \tag{3.16}$$

$$\frac{\partial q}{\partial t} + S\frac{\partial P}{\partial x} + \frac{2fc^2 q|q|}{DSP} + \frac{Sg\sin\theta}{c^2}P = 0, \tag{3.17}$$

Another set of assumptions comes from hypothesizing that the capacity of the gas duct is large and the boundary conditions remain relatively stable. With this consideration, the first term can also be neglected, yielding the following simplified equations:

$$\frac{\partial P}{\partial t} + \frac{c^2}{S}\frac{\partial q}{\partial x} = 0, \tag{3.18}$$

$$\frac{\partial P}{\partial x} = -\frac{2fc^2 q|q|}{DS^2 P} - \frac{g\sin\theta}{c^2}P, \tag{3.19}$$

As observed, the friction term in Eq. (3.19) uses the expression $q|q|$ instead of q^2, to ensure that the friction force opposes the direction of the gas movement. If there is no possibility of reverse flow in the network, q^2 can be retained. Applying a linearization technique to Eq. (3.19) and introducing the notation $\vec{\lambda} = \frac{|q|}{P}$, we obtain the following equations:

$$\frac{\partial P}{\partial t} + \frac{c^2}{S}\frac{\partial q}{\partial x} = 0 \tag{3.20}$$

$$\frac{\partial P}{\partial x} = -\frac{2f\vec{\lambda}c^2 q}{DS^2} - \frac{g\sin\theta}{c^2}P, \tag{3.21}$$

Differentiating Eq. (3.21) with respect to x yields:

$$\frac{\partial^2 P}{\partial x^2} = \frac{2f\vec{\lambda}c^2}{DS^2}\frac{\partial q}{\partial x} - \frac{g\sin\theta}{c^2}\frac{\partial P}{\partial x},$$

Substituting $\frac{\partial q}{\partial x} = -\frac{S}{c^2}\frac{\partial P}{\partial t}$ from Eq. (3.20) into the above equation, we get:

$$\frac{\partial^2 P}{\partial x^2} = -\frac{2f\vec{\lambda}}{Dc^2}\frac{\partial P}{\partial t} - \frac{g\sin\theta}{c^2}\frac{\partial P}{\partial x}. \tag{3.22}$$

Equation (3.22) can be rewritten in the form of a convective heat equation as follows:

$$\frac{\partial P}{\partial t} = a\frac{\partial^2 P}{\partial x^2} + b\frac{\partial P}{\partial x}, \tag{3.23}$$

where the parameters a and b are defined by:

$$a = -\frac{DS}{2f\vec{\lambda}}, \quad b = -\frac{DSg\sin\theta}{2f\vec{\lambda}c^2}. \tag{3.24}$$

Let us consider the following relation for pressure:

$$P(x,t) = U(x,t)\exp(\beta t + \mu x), \tag{3.25}$$

where β and μ are given by:

$$\beta = -\frac{b^2}{4a}, \quad \mu = -\frac{b}{2a}. \tag{3.26}$$

Substituting expression (3.24) into Eq. (3.23) results in the following simplified equation for U:

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2}, \tag{3.27}$$

This equation (3.25), subject to the constraint $0 \le x < L, \quad t \ge 0$, is solved with the initial condition:

$$U(x,0) = U_0(x), \tag{3.28}$$

and the boundary conditions:

$$U(0,t) = U_0(0,t), \quad \left.\frac{\partial U}{\partial x}\right|_{x=L} = 0. \tag{3.29}$$

The heat (diffusion) linear parabolic partial differential equation of the second order (3.25) has infinitely many partial solutions, and in particular, it admits solutions (see http://www.scholarpedia.org/article/Partial_differential_equation Partial Differential Equation on Scholarpedia):

$$U(x,t) = A\exp\left(-\alpha^2 t\right)\cos(\alpha x + B) + C, \tag{3.30}$$

where α, A, B, and C are arbitrary constants which are to be defined by the initial (3.26) and boundary (3.27) conditions.

An exact solution of the first order non-homogeneous heat (diffusion) linear parabolic partial differential equation (3.23), with the initial (3.26) and boundary (3.27) conditions, has the following form:

$$P(x,t) = A\exp\left(-\frac{b^2 t}{4a} - \frac{bx}{2a}\right)\left(\exp\left(-\alpha^2 t\right)\cos(\alpha x + B) + C\right), \tag{3.31}$$

Additionally, integrating equation (3.20) within the interval $[x, L]$ yields:

$$q(x,t) = q(L,t) - \frac{S}{c^2}\int_x^L \frac{\partial P}{\partial t}\,dx, \tag{3.32}$$

For theoretical and practical purposes, it is desirable to consider the movement of a gas mixture in an inclined branched pipeline and obtain some analytical solutions.

4. Gas flow in an inclined and branched pipeline

Let's consider the flow of gas within a branched and inclined pipeline, which represents a more common practical scenario. For isothermal gas flow, the dynamical laws of conservation of mass and momentum for such a pipeline are described by the following set of one-dimensional nonlinear partial differential equations (Davitashvili and Rukhaya, 2023):

$$\frac{\partial P}{\partial t} + \frac{c^2}{S}\frac{\partial q}{\partial x} + c^2 q^* \delta(x - x^*) = 0, \tag{4.1}$$

$$\frac{\partial q}{\partial t} + S\frac{\partial P}{\partial x} + \frac{2fc^2q^2}{DSP} + \frac{Sg\sin\theta}{c^2}P = 0, \tag{4.2}$$

where δ represents the Dirac delta function and x^* indicates the location of a branch in the pipeline. q^* signifies the volumetric gas consumption at a branch-line ($q^* = \mathrm{BV}, V$ is the volumetric rate of gas consumption in the branch-line at the offshoot).

The system of equations (4.1)-(4.2), with the constraints $0 \le x < L, t \ge 0$, is addressed by employing the following initial conditions:

$$P(x,0) = P_0(x), \quad q(x,0) = q_0(x), \tag{4.3}$$

and the boundary conditions:

$$P(0,t) = P_1(t), \quad q(L,t) = q_1(t), \tag{4.4}$$

Unfortunately, an analytical solution to the system of equations (4.1)-(4.2) with initial and boundary conditions (4.3)-(4.4) is not known, even for simpler pipeline configurations (i.e., when $q^* = 0$).

With the assumption that the boundary conditions vary slowly and there is no reverse flow in the network, employing a linearization technique with the notation $\vec{\lambda} = \frac{|q|}{P}$ and introducing $\Phi(x) = c^2 q^* \delta(x - x^*)$, we arrive at:

$$\frac{\partial P}{\partial t} + \frac{c^2}{S}\frac{\partial q}{\partial x} + \Phi(x) = 0, \tag{4.5}$$

$$\frac{\partial P}{\partial x} = -\frac{2f\vec{\lambda}c^2 q}{DS^2} - \frac{g\sin\theta}{c^2}P. \tag{4.6}$$

Differentiating Eq. (4.6) with respect to x yields:

$$\frac{\partial^2 P}{\partial x^2} = \frac{2f\vec{\lambda}c^2}{DS^2}\frac{\partial q}{\partial x} - \frac{g\sin\theta}{c^2}\frac{\partial P}{\partial x},$$

Now, substituting $\frac{\partial q}{\partial x} = -\frac{S}{c^2}\left(\Phi(x) + \frac{\partial P}{\partial t}\right)$ derived from Eq. (4.5) into the differential equation, we get:

$$\frac{\partial^2 P}{\partial x^2} = -\frac{2f\vec{\lambda}}{DS}\frac{\partial P}{\partial t} - \frac{g\sin\theta}{c^2}\frac{\partial P}{\partial x} + \frac{2f\vec{\lambda}}{DS}\Phi(x), \tag{4.7}$$

In reality, equation (4.7) represents a thermal conductivity equation with a source term:

$$\frac{\partial P}{\partial t} = a\frac{\partial^2 P}{\partial x^2} + b\frac{\partial P}{\partial x} + \Phi(x) \tag{4.8}$$

where

$$a = -\frac{DS}{2f\vec{\lambda}}, \quad b = -\frac{DSg\sin\theta}{2f\vec{\lambda}c^2}, \quad \Phi(x) = \frac{DSc^2q^*}{2f\vec{\lambda}}\delta(x - x^*).$$

By introducing the substitution (4.9)

$$P(x,t) = U(x,t)\exp(\beta t + \mu x), \tag{4.9}$$

where $\beta = -\frac{b^2}{4a}$ and $\mu = -\frac{b}{2a}$, we arrive at the following non-homogeneous equation:

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2} + \exp(-\beta t - \mu x)\Phi(x). \tag{4.10}$$

Solving equation (4.10) within the domain $0 \le x < L, \quad t \ge 0$, subject to the initial conditions

$$U(x,0) = U_0(x), \tag{4.11}$$

and boundary conditions

$$U(0,t) = U_0(0,t), \quad \frac{\partial U}{\partial x}|_{x=L} = 0, \tag{4.12}$$

we find the solution to the non-homogeneous heat equation as:

$$U(x,t) = \int_{-\infty}^{+\infty} U_0(\xi)G(x,\xi,t)d\xi + \int_0^t \int_{-\infty}^{+\infty} \Phi(\xi,\tau)G(x,\xi,\tau)d\xi d\tau,$$

where the Green's function is given by:

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi at}} \sum_{n=-\infty}^{+\infty} \left\{ \exp\left[-\frac{(x-\xi+2nL)^2}{4at}\right] + \exp\left[-\frac{(x+\xi+2nL)^2}{4at}\right]\right\}.$$

Finally, the exact solution of the considered second-order non-homogeneous heat equation (4.8) with initial and boundary conditions (4.11) and (4.12) is:

$$P(x,t) = \exp(\beta t + \mu x)\left[\int_{-\infty}^{+\infty} U_0(\xi)G(x,\xi,t)d\xi + \int_0^t \int_{-\infty}^{+\infty} \Phi(\xi,\tau)G(x,\xi,\tau)d\xi d\tau\right].$$

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