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The Functional Dissipativity of Linear Systems of PDEs: a Survey

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We survey some recent results obtained by Vladimir Maz'ya and myself concerning the functional dissipativity of second order systems of PDEs. In the particular case of operators of the form $\partial_h(\mathscr{A}^h(x)\partial_h u)$, where $\mathscr{A}^h$ are $m \times m$ matrices, we have given algebraic necessary and sufficient conditions. For such operators, we investigate the relations between different notions of functional ellipticity and functional dissipativity.

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1. Introduction

Let A be a scalar second order partial differential operator

$$Au = \operatorname{div}(\mathscr{A}\,\nabla u) \tag{1}$$

defined in a domain $\Omega \subset \mathbb{R}^N$. The coefficients a^{hk} are supposed to be essentially bounded complex valued functions ($a^{hk} \in L^\infty(\Omega)$).

Given a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, the operator (1) is said to be functional dissipative with respect to φ if

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(\varphi(|u|)u)\rangle\, dx \geqslant 0 \tag{2}$$

for any $u \in \mathring{H}^1(\Omega)$ such that $\varphi(|u|)u \in \mathring{H}^1(\Omega)$.

The concept of functional dissipativity was given in [13]. A motivation for the introduction of this notion will be given in Section 4.

When $\varphi(s) = s^{p-2}$ $(p > 1)$ we have the so-called L^p-dissipativity

$$\mathbb{R}\mathrm{e} \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(|u|^{p-2}u)\rangle\, dx \geqslant 0 \tag{3}$$

for any $u \in \mathring{H}^1(\Omega)$ such that $|u|^{p-2}u \in \mathring{H}^1(\Omega)$. In [7] necessary and sufficient condition for the validity of (3) have been obtained. In a series of papers the L^p-

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dissipativity has been considered for several different scalar and matrix partial differential operators (see [8, 9, 11, 12] and the monograph [10], where L^p-dissipative operators are considered in the more general frame of semi-bounded operators).

A strengthening of the condition given in [7] for the L^p-dissipativity has led to the concept of p-ellipticity, which is connected to the L^p solvability of the Dirichlet problem for operators with complex coefficients.

The concept of functional dissipativity for second order systems has been considered in [14, 15]. The present paper aims to survey the main results obtained by Vladimir Maz'ya and myself for a class of systems of PDEs. We mention that we have also obtained peculiar results for the functional dissipativity in linear elasticity, for which we refer to [15].

The present paper is organized as follows. After recalling the concept of L^p-dissipativity for scalar operators in Section 2, we describe the notion of p-ellipticity in Section 3 and present some applications of our results.

Section 4 is devoted to the functional dissipativity for scalar operators.

In Section 5 we consider the functional dissipativity for second order systems, giving necessary and sufficient conditions for a particular class of systems. We also give several notions of functional ellipticity and investigate the relations between them and the functional dissipativity.

2. L^p-dissipativity for scalar operators

Let A be the scalar second order operator (1). The following result provides a necessary and sufficient condition for its L^p-dissipativity (in all this paper we assume $1 < p < \infty$)

Theorem 2.1 [7] *Let the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ be symmetric, i.e. $\mathbb{I}\mathrm{m}\,\mathscr{A}^t = \mathbb{I}\mathrm{m}\,\mathscr{A}$. The operator A is L^p-dissipative if and only if*

$$|p-2|\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle \tag{4}$$

for almost any $x\in\Omega$ and for any $\xi\in\mathbb{R}^N$.

In [7] the result was proved considering more general operators with complex measures coefficients. We remark that from condition (4) you can immediately obtain some known results. Suppose that the operator A is such that

$$\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle \geqslant 0, \quad \text{a.e. } x\in\Omega,\ \forall\xi\in\mathbb{R}^N .$$

Then A is always L^2-dissipative. If A is a real coefficient operator, A is L^p-dissipative for any p.

We remark also that, if $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is not symmetric or if the operator A has lower order terms, then Theorem 2.1 does not hold. You can find some examples in [7].

The condition (4) is equivalent to the positivity of some polynomial in ξ and η. More exactly, (4) is equivalent to the following condition:

$$\frac{4}{p\,p'}\langle\, \mathbb{R}\mathrm{e}\,\mathscr{A}\xi,\xi\rangle + \langle\, \mathbb{R}\mathrm{e}\,\mathscr{A}\eta,\eta\rangle - 2(1-2/p)\langle\, \mathbb{I}\mathrm{m}\,\mathscr{A}\xi,\eta\rangle \geqslant 0 \tag{5}$$

for almost any $x\in\Omega$ and for any $\xi,\eta\in\mathbb{R}^N$.

More generally, if the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is not symmetric, condition (5) is still necessary for the L^p-dissipativity, but not sufficient. In this case, we can consider the condition

$$\frac{4}{p\,p'}\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\eta,\eta\rangle + 2\langle (p^{-1}\,\mathbb{I}\mathrm{m}\,\mathscr{A}(x) + p'^{-1}\,\mathbb{I}\mathrm{m}\,\mathscr{A}^*(x))\xi,\eta\rangle \geqslant 0 \quad (6)$$

for almost any $x \in \Omega$ and for any $\xi,\eta \in \mathbb{R}^N$ ($p' = p/(p-1)$). It turns out that (6) is sufficient for the L^p-dissipativity, but not necessary.

We mention that recently Maz'ya and Verbitsky [27, 28] gave necessary and sufficient conditions for the accretivity of a second order partial differential operator E containing lower order terms, in the case of Dirichlet data. We observe that the accretivity of E is equivalent to the L^2-dissipativity of $-E$.

3. *p*-ellipticity and applications of *L^p*-dissipativity

Let A be a scalar operator with lower order terms:

$$Au = \mathrm{div}(\mathscr{A}\,\nabla u) + b\nabla u + au\,. \quad (7)$$

The operator A is said to be p-elliptic if a strengthened version of inequality (6) holds. More precisely, A is p-elliptic if there exists $\kappa > 0$ such that

$$\begin{gathered}\frac{4}{p\,p'}\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\eta,\eta\rangle + \\ 2\left\langle \left(\frac{1}{p}\,\mathbb{I}\mathrm{m}\,\mathscr{A}(x) + \frac{1}{p'}\,\mathbb{I}\mathrm{m}\,\mathscr{A}^*(x)\right)\xi,\eta\right\rangle \geqslant \kappa(|\xi|^2 + |\eta|^2)\end{gathered}$$

a.e. $x \in \Omega$, $\forall\,\xi,\eta \in \mathbb{R}^N$.

Carbonaro and Dragičević [3, 4] showed that this condition implies some bilinear embeddings, i.e. the boundedness of certain bilinear operators arising from complex-valued second order differential operators. Their main result is the following

Theorem 3.1 [4] *Let $P_t^A = \exp(-tL_A)\,, t > 0$ and let $p > 1$. Suppose that the matrices A, B are p-elliptic. Then for all $f, g \in \mathring{C}^\infty(\mathbb{R}^N)$ we have*

$$\int_0^\infty \int_{\mathbb{R}^N} \left|\nabla P_t^A f(x)\right| \left|\nabla P_t^B g(x)\right| dxdt \leqslant C\|f\|_p \|g\|_{p'}\,, \quad (8)$$

with constant depending on ellipticity parameters, but not dimension.

If A and B are real accretive matrices then (8) holds for the full range of exponents $p \in (1,\infty)$.

Recently Carbonaro, Dragičević, Kovač and Škreb [5] extended this result to trilinear embeddings.

In a series of papers [17–20] Dindoš and Pipher proved several results concerning the L^p solvability of the Dirichlet problem

$$\begin{cases} \partial_i\left(a_{ij}(x)\partial_j u\right)+b_i(x)\partial_i u=0 & \text{in } \Omega \\ u(x)=f(x) & \text{a.e. on } \partial\Omega \\ \widetilde{N}_{2,a}(u)\in L^p(\partial\Omega) & \end{cases} \tag{9}$$

where f is in $L^p(\partial\Omega)$. Here $a>0$ is a fixed parameter and $\widetilde{N}_{2,a}(u)$ is a nontangential maximal function defined using L^p averages over balls

$$\widetilde{N}_{2,a}(u)(y)=\sup_{x\in\Gamma_a(y)}\left(\fint_{B_{\delta(x)/2}(x)}|u(z)|^2dz\right)^{1/2}$$

($y\in\partial\Omega$), where the barred integral indicates the average and $\Gamma_a(y)$ is a cone of aperture a. To be precise, they say that the Dirichlet problem (9) is solvable for a given $p\in(1,\infty)$ if there exists a $C=C(p,\Omega)>0$ such that for all complex-valued boundary data $f\in L^p(\partial\Omega)\cap\dot{B}^{2,2}_{1/2}(\partial\Omega)$ the unique "energy solution" satisfies the estimate

$$\left\|\widetilde{N}_{2,a}(u)\right\|_{L^p(\partial\Omega)}\leqslant C\|f\|_{L^p(\partial\Omega)}\,.$$

Since the space $\dot{B}^{2,2}_{1/2}(\partial\Omega)\cap L^p(\partial\Omega)$ is dense in $L^p(\partial\Omega)$ for each $p\in(1,\infty)$, there exists a unique continuous extension of the solution operator $f\mapsto u$ to the whole space $L^p(\partial\Omega)$, with u such that $\widetilde{N}_{2,a}(u)\in L^p(\partial\Omega)$ and the relevant estimate $\left\|\widetilde{N}_{2,a}(u)\right\|_{L^p(\partial\Omega)}\leqslant C\|f\|_{L^p(\partial\Omega)}$ is valid.

Egert [21] has shown that the p-ellipticity condition implies extrapolation to a holomorphic semigroup on Lebesgue spaces in a p-dependent range of exponents.

Our condition (6) and its strengthened variant are getting more and more important in many respects. We already considered the notion of p-ellipticity, but there are also other applications.

We mention that Hömberg, Krumbiegel and Rehberg [24] used some of the techniques introduced in [7] to show the L^p-dissipativity of a certain operator connected to the problem of the existence of an optimal control for the heat equation with dynamic boundary conditions.

Beyn and Otten [1, 2] considered the semilinear system

$$A\Delta v(x)+\langle Sx,\nabla v(x)\rangle+f(v(x))=0,\qquad x\in R^N,$$

where A is a $m\times m$ matrix, S is a $N\times N$ skew-symmetric matrix and f is a sufficiently smooth vector function. Among the assumptions they made, they require the existence of a constant $\gamma_A>0$ such that

$$|z|^2\,\mathbb{R}\mathrm{e}\langle w,Aw\rangle+(p-2)\,\mathbb{R}\mathrm{e}\langle w,z\rangle\,\mathbb{R}\mathrm{e}\langle z,Aw\rangle\geqslant\gamma_A|z|^2|w|^2$$

for any $z,w\in\mathbb{C}^m$. This condition originates from our necessary and sufficient condition for the L^p-dissipativity of certain systems (see [8, formula (79), p.261]).

The results of [7] allowed Nittka [29] to consider the case of partial differential operators with complex coefficients.

Ostermann and Schratz [30] have obtained the stability of a numerical procedure for solving a certain evolution problem. The necessary and sufficient condition (4) shows that their result does not require the contractivity of the corresponding semigroup.

Chill, Meinlschmidt and Rehberg [6] used some ideas from [7] in the study of the numerical range of second order elliptic operators with mixed boundary conditions in L^p.

ter Elst, Haller-Dintelmann, Rehberg and Tolksdorf [22] considered second order divergence form operators with complex coefficients, complemented with Dirichlet, Neumann, or mixed boundary conditions. They proved several results related to the generation of strongly continuous semigroups on L^p.

4. Functional dissipativity for scalar operators

Let Φ be a Young function, i.e. a convex positive function such that $\Phi(0)=0$ and $\Phi(+\infty)=+\infty$. Let us consider the Orlicz space of the function u such that there exists $\alpha>0$ such that

$$\int_\Omega \Phi(\alpha\,|u|)\,dx < +\infty\,.$$

The relevant Luxemburg norm is defined as

$$\|u\| = \inf\left\{\lambda>0 \;\Big|\; \int_\Omega \Phi(|u(x)|/\lambda)\,dx \leqslant 1\right\}.$$

For the general theory of Orlicz spaces see Krasnosel'skiĭ, Rutickiĭ [26] and Rao, Ren [31].

Let us consider the Cauchy problem

$$\begin{cases} u' = Au \\ u(0) = u_0\,. \end{cases} \tag{10}$$

where A is a certain linear operator. The condition for the decrease of the Luxemburg norm of solutions u of (10) is

$$\mathbb{R}\mathrm{e}\int_\Omega \langle Au,u\rangle |u|^{-1}\Phi'(|u|)\,dx \leqslant 0\,. \tag{11}$$

Indeed, at least formally, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \Phi(|u(x,t)|/\lambda)\,dx = \frac{1}{\lambda}\int_\Omega \Phi'(|u(x,t)|/\lambda)\,\frac{d}{dt}|u(x,t)|\,dx \\ &= \frac{1}{\lambda}\;\mathbb{R}\mathrm{e}\int_\Omega \langle u_t,u\rangle |u|^{-1}\Phi'(|u|)\,dx = \frac{1}{\lambda}\;\mathbb{R}\mathrm{e}\int_\Omega \langle Au,u\rangle |u|^{-1}\Phi'(|u|)\,dx\,. \end{aligned}$$

This shows that condition (11) implies the decrease of

$$\int_\Omega \Phi(|u(x,t)|/\lambda)\,dx$$

as t increases. It follows the decrease of the Luxemburg norm of solutions u of Cauchy problem (10).

Condition (11) can be written as

$$\Re\mathrm{e} \int_\Omega \langle Au, u\rangle\, \varphi(|u|)\,dx \leqslant 0 \tag{12}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$.

The relation between the functions φ and Φ is

$$\varphi(t) = \frac{\Phi'(t)}{t} \qquad \Longleftrightarrow \qquad \Phi(t) = \int_0^t s\,\varphi(s)\,ds\,.$$

Therefore the convexity of Φ (that we require in the definition of Orlicz space) is equivalent to the increase of the function $s\,\varphi(s)$. We note that we do not require the increase of φ. For example, if $\Phi(s) = s^p$ we have $\varphi(s) = p\,s^{p-2}$, and when $1 < p < 2$, φ is decreasing.

If A is the operator (1) and we make a formal integration by parts in (12), we get (2).

In [13] the functional dissipativity of operator (1) was introduced, under the following assumptions on φ:

(1) $\varphi \in C^1((0,+\infty))$;
(2) $(s\,\varphi(s))' > 0$ for any $s > 0$;
(3) the range of the strictly increasing function $s\,\varphi(s)$ is $(0,+\infty)$;
(4) there exist two positive constants C_1, C_2 and a real number $r > -1$ such that

$$C_1 s^r \leqslant (s\varphi(s))' \leqslant C_2\, s^r, \qquad s \in (0, s_0)$$

for a certain $s_0 > 0$. If $r = 0$ we require more restrictive conditions: there exists the finite limit $\lim_{s\to 0^+} \varphi(s) = \varphi_+(0) > 0$ and $\lim_{s\to 0^+} s\,\varphi'(s) = 0$.

(5) There exists $s_1 > s_0$ such that

$$\varphi'(s) \geqslant 0 \ \text{ or } \ \varphi'(s) \leqslant 0, \qquad \forall\ s \geqslant s_1.$$

The condition (4) prescribes the behaviour of the function φ in a neighborhood of the origin, while (5) concerns the behaviour for large s.

The function $\Phi(s) = s^p$, i.e. $\varphi(s) = s^{p-2}$ $(p > 1)$ provides an example of such a function. Other noteworthy examples are given by the Young function corresponding to the Zygmund space $L^p \log L$, $\Phi(s) = s^p \log(s + e)$ $(p > 1)$, i.e. $\varphi(s) = ps^{p-2}\log(s+e) + s^{p-1}(s+e)^{-1}$, and $\Phi(s) = \exp(s^p) - 1$, i.e. $\varphi(s) = p\,s^{p-2}\exp(s^p)$.

Necessary and sufficient conditions for the functional dissipativity of the operator (1) have been obtained in [13]. Our main result is the following

Theorem 4.1 [13] *Let the matrix* $\mathbb{I}\mathrm{m}\,\mathscr{A}$ *be symmetric, i.e.* $\mathbb{I}\mathrm{m}\,\mathscr{A}^t = \mathbb{I}\mathrm{m}\,\mathscr{A}$. *Then the operator* (1) *is* L^Φ*-dissipative if, and only if,*

$$|s\,\varphi'(s)|\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\,\xi,\xi\rangle| \leqslant 2\sqrt{\varphi(s)\,[s\,\varphi(s)]'}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\xi,\xi\rangle \tag{13}$$

for almost every $x\in\Omega$ *and for any* $s>0, \xi\in\mathbb{R}^N$.

We have also

Corollary 4.2 [13] *Let the matrix* $\mathbb{I}\mathrm{m}\,\mathscr{A}$ *be symmetric, i.e.* $\mathbb{I}\mathrm{m}\,\mathscr{A}^t = \mathbb{I}\mathrm{m}\,\mathscr{A}$. *If*

$$\lambda_0 = \sup_{s>0} \frac{|s\,\varphi'(s)|}{2\sqrt{\varphi(s)\,[s\,\varphi(s)]'}} < +\infty,$$

then the operator (1) *is* L^Φ*-dissipative if, and only if,*

$$\lambda_0\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\,\xi,\xi\rangle| \leqslant \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\xi,\xi\rangle \tag{14}$$

for almost every $x\in\Omega$ *and for any* $\xi\in\mathbb{R}^N$. *If* $\lambda_0 = +\infty$ *the operator* (1) *is* L^Φ*-dissipative if and only if* $\mathbb{I}\mathrm{m}\,\mathscr{A}\equiv 0$ *and*

$$\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle \geqslant 0 \tag{15}$$

for almost every $x\in\Omega$ *and for any* $\xi\in\mathbb{R}^N$.

We remark that λ_0 may be finite or not. For example, if the Orlicz space is L^p or the Zygmund space $L^p\log L$ (i.e. $\Phi(s)=s^p$ or $\Phi(s)=s^p\log(s+e)$ $(1<p<\infty)$), then λ_0 is finite and the operator A is is L^Φ-dissipative if and only if (14) holds. If the Orlicz space is the one related to $\Phi(s)=\exp(s^p)-1$, then $\lambda_0=+\infty$ and the operator A is L^Φ-dissipative if and only if $\mathbb{I}\mathrm{m}\,\mathscr{A}$ identically vanish and (15) holds.

As for L^p-dissipativity, condition (13) is equivalent to the positivity of some polynomial in ξ and η. To describe such results, let us introduce the function Λ, which is defined by the relation

$$\Lambda\left(s\sqrt{\varphi(s)}\right) = -\frac{s\,\varphi'(s)}{s\,\varphi'(s)+2\,\varphi(s)}\,.$$

Assuming that $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is symmetric, we have that (13) is equivalent to the following condition:

$$[1-\Lambda^2(t)]\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\eta,\eta\rangle + 2\,\Lambda(t)\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\,\xi,\eta\rangle \geqslant 0 \tag{16}$$

for almost every $x\in\Omega$ and for any $t>0, \xi,\eta\in\mathbb{R}^N$ (see [13, Remark 2, p.23]).

If the condition $\mathbb{I}\mathrm{m}\,\mathscr{A} = \mathbb{I}\mathrm{m}\,\mathscr{A}^t$ is not satisfied, condition (13) is still necessary for the L^Φ-dissipativity of the operator A, but in general it is not sufficient, whatever the function φ may be (see [13, p.23]).

If $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is not symmetric, we have that, if

$$\begin{gathered}[1-\Lambda^2(t)]\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\eta,\eta\rangle + \\ [1+\Lambda(t)]\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\,\xi,\eta\rangle + [1-\Lambda(t)]\langle \mathbb{I}\mathrm{m}\,\mathscr{A}^*(x)\,\xi,\eta\rangle \geqslant 0\end{gathered} \tag{17}$$

for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^N$, then the operator (1) is L^Φ-dissipative. In general, condition (17) is not necessary for the L^Φ-dissipativity of A. Note that, if $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is symmetric, then (17) coincides with (16).

In [13] the concept of functional ellipticity (Φ-(strong) ellipticity) was introduced as well. If the principal part of the operator (7) is such that the left-hand side of (17) is not merely non negative but strictly positive, i.e. there exists $\kappa > 0$ such that

$$[1-\Lambda^2(t)]\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\,\eta,\eta\rangle +$$
$$[1+\Lambda(t)]\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\,\xi,\eta\rangle + [1-\Lambda(t)]\langle \mathbb{I}\mathrm{m}\,\mathscr{A}^*(x)\,\xi,\eta\rangle \geqslant \kappa(|\xi|^2+|\eta|^2)$$

for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^N$, we say that the operator A is Φ-elliptic.

Very recently Kovač and Škreb [25] proved bilinear embeddings in Orlicz spaces by using conditions introduced in [13].

5. Second order systems

In [8] we have considered the L^p-dissipativity for second order systems, obtaining several criteria. For all the details we refer to [8].

In this section, we describe some of the recent results we have obtained concerning the functional dissipativity of systems in divergence form. In the first subsection, we discuss the notion of functional dissipativity and give necessary and sufficient conditions for a particular class of systems. In the other subsection, we investigate different notions of functional ellipticity.

5.1. *Functional dissipativity*

Let us consider a general system of the form

$$A = \partial_h(\mathscr{A}^{hk}(x)\partial_k) \tag{18}$$

where $\partial_k = \partial/\partial x_k$ and $\mathscr{A}^{hk}(x) = \{a_{ij}^{hk}(x)\}$ are $m \times m$ matrices whose elements are complex valued L^1_{loc}-functions defined in a domain $\Omega \subset \mathbb{R}^N$ $(1 \leqslant i,j \leqslant m,\ 1 \leqslant h,k \leqslant N)$. We say that the operator (18) is L^Φ-dissipative if

$$\mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{A}^{hk}\,\partial_k u, \partial_h(\varphi(|u|)\,u)\rangle\,dx \geqslant 0$$

for any $u \in [\mathring{C}^1(\Omega)]^m$ such that $\varphi(|u|)\,u \in [\mathring{C}^1(\Omega)]^m$.

We say that the operator (18) is strict L^Φ-dissipative if there exists $\kappa > 0$ such that

$$\mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{A}^{hk}\,\partial_k u, \partial_h(\varphi(|u|)\,u)\rangle\,dx \geqslant \kappa\int_\Omega |\nabla(\sqrt{\varphi(|u|)}\,u)|^2 dx$$

for any $u \in [\mathring{C}^1(\Omega)]^m$ such that $\varphi(|u|)\,u \in [\mathring{C}^1(\Omega)]^m$.

In the particular case of matrix operators defined as

$$Au = \partial_h(\mathscr{A}^h(x)\partial_h u) \tag{19}$$

where $\mathscr{A}^h(x) = \{a^h_{ij}(x)\}$ $(i, j = 1, \dots, m)$ are matrices with complex locally integrable entries $(h = 1, \dots, N)$, we have found necessary and sufficient conditions.

From now on we require also the following condition on φ:

(6) the function $|s\,\varphi'(s)/\varphi(s)|$ is not decreasing.

This implies that the function $\Lambda^2(t)$ is not decreasing on $(0, +\infty)$ (see [15, Lemma 8]). Since the function Λ does not change the sign, we have the monotonicity of the bounded function $\Lambda(t)$ and then the existence of the finite limit

$$\Lambda_\infty = \lim_{t\to+\infty} \Lambda(t)\,.$$

We have also

$$\Lambda^2_\infty = \sup_{t>0} \Lambda^2(t)\,.$$

We are now in a position to describe the aforesaid necessary and sufficient conditions

Theorem 5.1 [14] *The operator A is L^Φ-dissipative if and only if*

$$\mathbb{R}\mathrm{e}\langle\mathscr{A}^h(x_0)\lambda, \lambda\rangle - \Lambda^2_\infty\,\mathbb{R}\mathrm{e}\langle\mathscr{A}^h(x_0)\omega, \omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda, \omega\rangle)^2$$
$$+\Lambda_\infty\,\mathbb{R}\mathrm{e}(\langle\mathscr{A}^h(x_0)\omega, \lambda\rangle - \langle\mathscr{A}^h(x_0)\lambda, \omega\rangle)\,\mathbb{R}\mathrm{e}\langle\lambda, \omega\rangle \geqslant 0$$

holds for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, N$.

A similar result concerns the strict functional dissipativity

Theorem 5.2 [14] *Let us assume that*

$$\Lambda^2_\infty < 1. \tag{20}$$

The operator (19) *is strict L^Φ-dissipative if and only if there exists $\kappa' > 0$ such that*

$$\mathbb{R}\mathrm{e}\langle\mathscr{A}^h(x_0)\lambda, \lambda\rangle - \Lambda^2_\infty\,\mathbb{R}\mathrm{e}\langle\mathscr{A}^h(x_0)\omega, \omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda, \omega\rangle)^2$$
$$+\Lambda_\infty\,\mathbb{R}\mathrm{e}(\langle\mathscr{A}^h(x_0)\omega, \lambda\rangle - \langle\mathscr{A}^h(x_0)\lambda, \omega\rangle)\,\mathbb{R}\mathrm{e}\langle\lambda, \omega\rangle \geqslant \kappa'\,|\lambda|^2$$

holds for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, N$.

5.2. *Functional ellipticity*

It is well known that there are different notions of ellipticity for systems. Indeed let us consider again the system (18), where $\mathscr{A}^{hk}(x) = \{a^{hk}_{ij}(x)\}_{m\times m}$, $a^{hk}_{ij} \in L^\infty(\Omega)$ $(1 \leqslant i, j \leqslant m,\ 1 \leqslant h, k \leqslant N)$. We have at least the following three notions.

The operator A is said to be *strong elliptic* if the following Legendre condition is satisfied

$$\mathbb{R}\mathrm{e}\langle \mathscr{A}^{hk}(x)\zeta_k, \zeta_h\rangle \geqslant \kappa|\zeta|^2, \text{ a.e. } x \in \Omega, \forall\, \zeta_h \in \mathbb{C}^m.$$

If the integral condition

$$\mathbb{R}\mathrm{e}\int_\Omega \langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle\, dx \geqslant \kappa \int_\Omega |\nabla v|^2 dx \tag{21}$$

holds for any $v \in \mathring{H}^1(\Omega)$, we have the so-called *integral ellipticity.*

Finally, if the Legendre–Hadamard condition

$$\mathbb{R}\mathrm{e}\langle(\mathscr{A}^{hk}(x)q_h q_k)\lambda, \lambda\rangle \geqslant \kappa\,|q|^2|\lambda|^2 \tag{22}$$

holds a. e. in Ω, for all $q \in \mathbb{R}^N, \lambda \in \mathbb{C}^m$, we have the so-called *weak ellipticity.*

In all these conditions κ is a positive constant.

It is clear that strong ellipticity implies integral ellipticity and it is well known that integral ellipticity implies the Legendre–Hadamard condition (just take $v = \lambda\, e^{i\, q\cdot x}\psi(x)$ in (21), ψ being a test function; see, e.g., [23, p.107]). If the operator is a real scalar operator ($N = 1$), then the three conditions are equivalent, but in general they are not. We recall that (22) implies the Gårding inequality, which is more general than (21) (see, e.g., [23, p.102]).

In [16] Dindoš, Li and Pipher gave three kinds of p-ellipticity for second order elliptic systems, which generalize the concepts of strong, integral, and weak ellipticity.

In [14] we have further extended these concepts within the theory of functional dissipativity. These are the precise definitions for the operator (18).

We say that the tensor $\{a^{hk}_{ij}(x)\}$ satisfies the strong Φ-ellipticity condition if there exists $\kappa > 0$ such that

$$\begin{gathered}\mathbb{R}\mathrm{e}\langle \mathscr{A}^{hk}(x)\xi_k, \xi_h\rangle - \Lambda^2(t)\,\mathbb{R}\mathrm{e}\langle(\mathscr{A}^{hk}(x) - (\mathscr{A}^{kh})^*(x))\omega, \xi_h\rangle\,\mathbb{R}\mathrm{e}\langle\omega, \xi_k\rangle \\ +\Lambda(t)\langle \mathscr{A}^{hk}(x)\omega, \omega\rangle\,\mathbb{R}\mathrm{e}\langle\omega, \xi_k\rangle\,\mathbb{R}\mathrm{e}\langle\omega, \xi_h\rangle \geqslant \kappa|\xi|^2\end{gathered} \tag{23}$$

a. e. in Ω, for any $\xi_h, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $t > 0$.

We say that the tensor $\{a^{hk}_{ij}(x)\}$ satisfies the integral Φ-ellipticity condition if there exists $\kappa > 0$ such that

$$\begin{gathered}\mathbb{R}\mathrm{e}\int_\Omega \Big(\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle + \Lambda(|v|)\,|v|^{-2}\langle\Big(\mathscr{A}^{hk} - (\mathscr{A}^{kh})^*\Big)\, v, \partial_h v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle \\ -\Lambda^2(|v|)\,|v|^{-4}\langle \mathscr{A}^{hk}\, v, v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_k v\rangle\,\mathbb{R}\mathrm{e}\langle v, \partial_h v\rangle\Big) dx \geqslant \kappa \int_\Omega |\nabla v|^2 dx\end{gathered} \tag{24}$$

holds for any $v \in [\mathring{C}^1(\Omega)]^m$. We note that if there are no lower order terms, as in the case we are considering here, the concepts of strong dissipativity and integral ellipticity are equivalent, thanks to Lemma 2.2 in [14, p.294]. This is not the case if there are lower order terms. The Φ-ellipticity is still given by (24), while the formula for Φ-dissipativity has to be changed, taking into account the lower order terms.

As in the classical case, (23) implies that the integrand in (24) is non-negative almost everywhere and therefore strong Φ-ellipticity implies integral Φ-ellipticity.

We say that the tensor $\{a_{ij}^{hk}(x)\}$ satisfies the Legendre-Hadamard Φ-ellipticity condition (or weak Φ-ellipticity condition) if there exixts $\kappa > 0$ such that

$$\begin{gathered}\mathbb{R}\mathrm{e}\langle(\mathscr{A}^{hk}(x)q_h q_k)\lambda,\lambda\rangle - \Lambda^2(t)\,\mathbb{R}\mathrm{e}\langle(\mathscr{A}^{hk}(x)q_h q_k)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2\\ +\Lambda(t)\,\mathbb{R}\mathrm{e}(\langle(\mathscr{A}^{hk}(x)q_h q_k)\omega,\lambda\rangle - \langle(\mathscr{A}^{hk}(x)q_h q_k)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle\\ \geqslant \kappa\,|q|^2|\lambda|^2\end{gathered} \tag{25}$$

a. e. in Ω, for any $q \in \mathbb{R}^N$, $\lambda,\ \omega \in \mathbb{C}^m$, $|\omega| = 1$, $t > 0$.

If $\varphi(t) = t^{p-2}$ and then $\Lambda(t) = -(1-2/p)$, conditions (23), (24), and (25) coincide with (17), (20), and (31) of [16], respectively.

We remark that, if condition (20) holds true, then inequalities (23) and (25) for any $t > 0$ are equivalent to

$$\begin{gathered}\mathbb{R}\mathrm{e}\langle\mathscr{A}^{hk}(x)\xi_k,\xi_h\rangle - \Lambda_\infty^2\,\mathbb{R}\mathrm{e}\langle(\mathscr{A}^{hk}(x) - (\mathscr{A}^{kh})^*(x))\omega,\xi_h\rangle\,\mathbb{R}\mathrm{e}\langle\omega,\xi_k\rangle\\ +\Lambda_\infty\langle\mathscr{A}^{hk}(x)\omega,\omega\rangle\,\mathbb{R}\mathrm{e}\langle\omega,\xi_k\rangle\,\mathbb{R}\mathrm{e}\langle\omega,\xi_h\rangle \geqslant \kappa|\xi|^2\end{gathered}$$

and

$$\begin{gathered}\mathbb{R}\mathrm{e}\langle(\mathscr{A}^{hk}(x)q_h q_k)\lambda,\lambda\rangle - \Lambda_\infty^2\,\mathbb{R}\mathrm{e}\langle(\mathscr{A}^{hk}(x)q_h q_k)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2\\ +\Lambda_\infty\,\mathbb{R}\mathrm{e}(\langle(\mathscr{A}^{hk}(x)q_h q_k)\omega,\lambda\rangle - \langle(\mathscr{A}^{hk}(x)q_h q_k)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle\\ \geqslant \kappa\,|q|^2|\lambda|^2,\end{gathered}$$

respectively (see [14]).

These concepts are interesting also in the case $N = 1$, when A is the ordinary differential operator

$$Au = (\mathscr{A}(x)u')', \tag{26}$$

$\mathscr{A}(x) = \{a_{ij}(x)\}$ $(i,j = 1,\dots,m)$ being a matrix with complex locally integrable entries defined in the bounded or unbounded interval $(a,b) \subset \mathbb{R}$.

It is natural to ask what are the relations between the functional dissipativity and the different notions of functional ellipticity for the operator (19). Indeed we have

Theorem 5.3 [14] *Let $N = 1$ and A be the operator* (26). *Assume* (20) *holds. The following statements are equivalent:*

(a) the operator A is strict L^Φ-dissipative;
(b) there exists $\kappa > 0$ such that $A - kI(d^2/dx^2)$ is L^Φ-dissipative;
(c) the matrix $\{a_{ij}(x)\}$ satisfies the strong Φ-ellipticity condition;
(d) the matrix $\{a_{ij}(x)\}$ satisfies the integral Φ-ellipticity condition;
(e) the matrix $\{a_{ij}(x)\}$ satisfies the weak Φ-ellipticity condition.

A slightly different result holds in higher dimensions for the operator (19).

Theorem 5.4 [14] *Let $N \geqslant 2$ and A be the operator* (19). *Assume* (20) *holds.*

The following statements are equivalent:

(a) the operator A is strict L^{Φ}-dissipative;
(b) there exists $\kappa > 0$ such that $A - k\Delta$ is L^{Φ}-dissipative;
(c) the matrix $\{a_{ij}(x)\}$ satisfies the integral Φ-ellipticity condition;
(d) the matrix $\{a_{ij}(x)\}$ satisfies the weak Φ-ellipticity condition.

Moreover, if the matrix $\{a_{ij}(x)\}$ satisfies the strong Φ-ellipticity condition, then (a)-(d) hold.

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