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A Dynamical Problem of Zero Approximation of Hierarchical Models for Fluids

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We investigate dynamical problem of zero approximation of hierarchical models for the Stokes case of incompressible viscous fluids. 2D problem then fluid occupied the domain of constant thickens and 1D problem then fluid occupied the domain of variable thickens vanishing at the part of the boundary are investigated.

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Introduction

We investigate dynamical problem of zero approximation of hierarchical models for fluids (see, e.g. [1], [2] and references therein) which is occupies $3D$ bounded region $\bar{\Omega}$ with boundary $\partial\Omega$:

$$\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x := (x_1, x_2) \in \omega, \ \overset{(-)}{h} < x_3 < \overset{(+)}{h} \right\},$$

where $\bar{\omega} = \omega \cup \partial\omega$ is the so-called projection of the plane $\bar{\Omega} = \Omega \cup \partial\Omega$.

In that follows, we assume that $\overset{(\pm)}{h} \in C^2(\omega) \cap C(\bar{\omega})$, $2h$ is the thickness of the $\bar{\Omega}$:

$$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x) > 0 \ \text{ for } \ x \in \omega$$

and

$$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x) \geq 0 \ \text{ for } \ x \in \partial\omega,$$

i.e., the thickness may vanish on some part of the boundary.

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As is well known, motion of the Newtonian fluid is characterized by the following equations

$$\rho\frac{dv_i(x_1,x_2,x_3,t)}{dt}=\sigma_{ji,j}(x_1,x_2,x_3,t)+\Phi_i(x_1,x_2,x_3,t),\quad i=\overline{1,3},$$
$$\sigma_{ji}=-\delta_{ji}p+\lambda\delta_{ji}\vartheta(v)+2\mu\epsilon_{ji}(v),\quad i,j=\overline{1,3},$$
$$\epsilon_{ji}(v):=\frac{1}{2}\Big(v_{j,i+v_{i,j}}\Big),\quad i,j=\overline{1,3},$$
$$\vartheta:=\epsilon_{ii}=v_{k,k}=:\mathrm{div}v,$$
$$\lambda:=\mu'-\frac{2}{3}\mu$$

where $v:=(v_1,v_2,v_3)$ is a velocity vector, σ_{ij} is a stress tensor, ϵ_{ij} is a velocity tensor, p is a pressure, Φ_i, $i=\overline{1,3}$, are components of the volume force, μ is the viscosity, μ' is the second viscosity, ρ is a density of the fluid. Throughout the paper we use, on the one hand Einstein's summation convention on repeated indices.

In the $N=0$ approximations of hierarchical models for fluids we get the following governing equations (see, [1], [2])

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta}+\Big[\lambda h\overset{0}{\widetilde{v}}_{\gamma0,\gamma}\Big]_{,\beta}+\Big[\mu h\Big(\overset{0}{\widetilde{v}}_{\alpha0,\beta}+\overset{0}{\widetilde{v}}_{\beta0,\alpha}\Big)\Big]_{,\alpha}+\overset{0}{X}_\beta(x_1,x_2,t)=\rho h\frac{\partial\overset{0}{\widetilde{v}}_{\beta0}}{\partial t},\quad\beta=1,2\tag{1}$$

$$\Big(\mu h\overset{0}{\widetilde{v}}_{30,\alpha}\Big)_{,\alpha}+\overset{0}{X}_3(x_1,x_2,t)=\rho h\frac{\partial\overset{0}{\widetilde{v}}_{30}}{\partial t},\tag{2}$$

where

$$0\le x_\beta\le l_\beta,\ l_\beta=const>0,\quad\beta=1,2,$$
$$\overset{0}{\widetilde{v}}_{j0}(x_1,x_2,t):=\frac{v_{j0}(x_1,x_2,t)}{h(x_2)},\quad\overset{0}{\widetilde{p}}_0(x_1,x_2,t):=\frac{p_r(x_1,x_2,t)}{h(x_2)},\tag{3}$$

are so called zeroth weighted moments of the velocity vector components and pressure, correspondingly,

The stress vector components on the upper and lower face surfaces be assumed to be known

$$\overset{0}{X}_i:=Q_{\overset{(+)}{n}i}\sqrt{1+\Big(\overset{(+)}{h}_{,1}\Big)^2+\Big(\overset{(+)}{h}_{,2}\Big)^2}+Q_{\overset{(-)}{n}i}\sqrt{1+\Big(\overset{(-)}{h}_{,1}\Big)^2+\Big(\overset{(-)}{h}_{,2}\Big)^2}+\Phi_{i0},\tag{4}$$
$$i=\overline{1,3}.$$

We naturally use known values of stress vector components $Q_{\overset{(+)}{n}i}$, $Q_{\overset{(-)}{n}i}$ on the face surfaces and Fourier-Legendre expansions of velocities there. Φ_{i0} is the zeroth order moment of the volume forces.

In the case of incompressible barotropic fluids, to the Navier-Stokes equations we add the equation

$$\mathrm{div} v = 0$$

which in terms of weighted moments (3) has a form (see [2])

$$\overset{0}{\widetilde{v}}_{\gamma 0,\gamma} = 0. \tag{5}$$

In view of (5) equation (1) can be rewritten as follows

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta} + \Big[\mu h\Big(\overset{0}{\widetilde{v}}_{\alpha 0,\beta} + \overset{0}{\widetilde{v}}_{\beta 0,\alpha}\Big)\Big]_{,\alpha} + \overset{0}{X}_\beta = \rho h \frac{\partial \overset{0}{\widetilde{v}}_{\beta 0}}{\partial t}, \quad \beta = 1, 2. \tag{6}$$

1. Two dimensional case for $2h = const$

In case of $2h = const$ if we differentiate first equation of (6) with respect to x_1 the second equations with respect to x_2, after summations of the obtained equations and taking into account of (5) for $\overset{0}{\widetilde{p}}_0(\cdot, t) \in C^2(\omega) \cap C(\bar{\omega})$ we get the following problem

$$\Delta_2 \overset{0}{\widetilde{p}}_0 = -\overset{0}{X}_{\alpha,\alpha}/h \tag{7}$$

$$\begin{aligned}
&\overset{0}{\widetilde{p}}_0(0, x_2, t) = p_0^1(x_2, t) \;\; \overset{0}{\widetilde{p}}_0(l_1, x_2, t) = p_l^1(x_2, t),\\
&\overset{0}{\widetilde{p}}_0(x_1, 0, t) = p_0^2(x_1, t) \;\; \overset{0}{\widetilde{p}}_0(x_1, l_2, t) = p_l^2(x_1, t),\\
&p_0^1(0, t) = p_0^2(0, t), \;\; p_l^1(l_2, t) = p_0^1(l_1, t),
\end{aligned} \tag{8}$$

by Δ_2 we denote 2D Laplace operator.

System (6), (2) can be rewritten as follows

$$\mu \Delta_2 \overset{0}{\widetilde{v}}_{i0} + F_i = \rho \frac{\partial \overset{0}{\widetilde{v}}_{i0}}{\partial t}, \quad i = 1, 2, 3, \tag{9}$$

where

$$F_i := (\overset{0}{\widetilde{p}}_0)_{,i} + \frac{\overset{0}{X}_i}{h}, \quad i = 1, 2, 3.$$

Problem 1. Let for simplicity $F_i \equiv 0$. Find $\overset{0}{\widetilde{v}}_{i0}, \;\; i = 1, 2, 3$ with

$$\begin{aligned}
&\overset{0}{\widetilde{v}}_{i0}(\cdot, t) \in C^2(\omega) \cap C(\bar{\omega}),\\
&\overset{0}{\widetilde{v}}_{i0}(x, \cdot) \in C(t \geq 0) \cap C^1(t > 0),\\
&\overset{0}{\widetilde{v}}_{i0}(x, t) \in C(\bar{\omega}, t \geq 0), \quad i = 1, 2, 3,
\end{aligned}$$

satisfying equations (9), boundary

$$\overset{0}{\widetilde{v}}_{i0}(0,x_2,t)=\overset{0}{\widetilde{v}}_{i0}(l_1,x_2,t)=0,\ \ i=1,2,3, \tag{10}$$

$$\overset{0}{\widetilde{v}}_{i0}(x_1,0,t)=\overset{0}{\widetilde{v}}_{i0}(x_1,l_2,t)=0,\ \ i=1,2,3, \tag{11}$$

conditions and initial

$$\overset{0}{\widetilde{v}}_{i0}(x_1,x_2,0)=f_i(x_1,x_2),\ \ 0\le x_\alpha\le l_\alpha,\ \ \alpha=1,2,\ \ i=1,2,3, \tag{12}$$

conditions, where

$$f_i(x_1,x_2)\in C^2(\omega)\cap C(\bar{\omega}),\ \ i=1,2,3,$$

are given functions, satisfy the following conditions

$$f_i(0,x_2)=f_i(l_1,x_2)=f_i(x_1,0)=f_i(x_1,l_2)=0,\ \ i=1,2,3.$$

Looking for solutions in the form as follows

$$\overset{0}{\widetilde{v}}_{i0}(x_1,x_2,t)=X_{1i}(x_1)X_{2i}(x_2)T_i(t),\ \ i=1,2,3,$$

plugging them into (9) and using the boundary conditions (11) yields the separated boundary value problems:

$$X''_{1i}(x_1)-B_iX_{1i}(x_1)=0,\ \ X_{1i}(0)=X_{2i}(l_1)=0,\ \ i=1,2,3, \tag{13}$$

$$X''_{2i}(x_2)-C_iX_{ii}(x_2)=0,\ \ X_{2i}(0)=X_{2i}(l_2)=0,\ \ i=1,2,3, \tag{14}$$

$$T'_i(t)-\sqrt{\frac{\mu}{\rho}}(B_i+C_i)T_i(t)=0,\ \ i=1,2,3, \tag{15}$$

where B_i and C_i $(i=1,2,3)$ are positive constants. Using the following representations of solutions (13)-(15)

$$X_{1im}=\sin\frac{m_i\pi}{l_1}x_1,\ \ B_{im}=-\Big(\frac{m_i\pi}{l_1}\Big)^2$$

$$X_{2in}=\sin\frac{n_i\pi}{l_2}x_2,\ \ C_{in}=-\Big(\frac{n_i\pi}{l_2}\Big)^2$$

$$T_{m_in_i}(t)=e^{-\lambda^2_{m_in_i}t}$$

where

$$\lambda_{m_in_i}=\sqrt{\frac{\mu}{\rho}}\pi\sqrt{\Big(\frac{m_i}{l_1}\Big)^2+\Big(\frac{n_i}{l_2}\Big)^2}$$

general solutions of (9)-(11) can be expressed by the following series

$$\overset{0}{\widetilde{v}}_{i0} = \sum_{m_i=1}^{\infty} \sum_{n_i=1}^{\infty} A_{im_in_i} \sin\Big(\frac{m_i\pi}{l_1}x_1\Big) \sin\Big(\frac{m_i\pi}{l_2}x_2\Big) e^{-\lambda^2_{m_in_i}t}.$$

Now, we should determine the values of the coefficients $A_{im_in_i}$ so that our solutions also satisfy the initial conditions (12). We need

$$f_i(x_1,x_2) = \overset{0}{\widetilde{v}}_{i0}(x_1,x_2,0) = \sum_{m_i=1}^{\infty} \sum_{n_i=1}^{\infty} A_{im_in_i} \sin\Big(\frac{m_i\pi}{l_1}x_1\Big) \sin\Big(\frac{m_i\pi}{l_2}x_2\Big),$$

which is just the double Fourier series for $f_i(x_1,x_2)$. We know that if $f_i \in C^2$ functions then

$$A_{im_in_i} = \frac{4}{l_1l_2}\int_0^{l_1}\int_0^{l_2} f_i(x_1,x_2) \sin\Big(\frac{m_i\pi}{l_1}x_1\Big) \sin\Big(\frac{m_i\pi}{l_2}x_2\Big) dx_1dx_2.$$

Theorem 1.1: *Suppose that $f(x_1,x_2) \in C^2([0,l_1]\times[0,l_2])$. The solutions of the Problem 1 can be found as absolutely and uniformly convergent series*

$$\overset{0}{\widetilde{v}}_{i0} = \sum_{m_i=1}^{\infty} \sum_{n_i=1}^{\infty} A_{im_in_i} \sin\Big(\frac{m_i\pi}{l_1}x_1\Big) \sin\Big(\frac{m_i\pi}{l_2}x_2\Big) e^{-\lambda^2_{m_in_i}t}$$

where

$$A_{im_in_i} = \frac{4}{l_1l_2}\int_0^{l_1}\int_0^{l_2} f_i(x_1,x_2) \sin\Big(\frac{m_i\pi}{l_1}x_1\Big) \sin\Big(\frac{m_i\pi}{l_2}x_2\Big) dx_1dx_2.$$

2. One dimensional case for the variable thickness vanishing at the boundary

Let

$$2h(x_1,x_2) = h_0x_2^{\alpha}, \ \ x_2 \in [0,l_2] \ \ h_0, \ l_2 \ \alpha = const > 0, \ \ 0<\alpha<1.$$

We consider case then all mechanical quantities in the system of Navier-Stokes equations are independent on x_1. After constructed zero approximation of hierarchical models for incompressible fluids we get the following system

$$\Big[\mu h(x_2)\overset{0}{\widetilde{v}}_{10,2}(x_2,t)\Big]_{,2} + \overset{0}{X}_1(x_2,t) = \rho h(x_2)\frac{\partial \overset{0}{\widetilde{v}}_{10}(x_2,t)}{\partial t}, \tag{16}$$

$$\Big(\mu h(x_2)\overset{0}{\widetilde{v}}_{30,2}(x_2,t)\Big)_{,2} + \overset{0}{X}_3(x_2,t) = \rho h(x_2)\frac{\partial \overset{0}{\widetilde{v}}_{30}(x_2,t)}{\partial t}, \tag{17}$$

$$(h(x_2)\overset{0}{\widetilde{p}}_0(x_2,t))_{,2} + \Big[\mu h(x_2)\overset{0}{\widetilde{v}}_{10,2}(x_2,t)\Big]_{,2} + \overset{0}{X}_2(x_2,t) = 0, \tag{18}$$

which we solve under boundary conditions (BCs)

$$\overset{0}{\widetilde{v}}_{10}(0,t)=0,\ \ \overset{0}{\widetilde{v}}_{10}(l_2,t)=0, \tag{19}$$

$$\overset{0}{\widetilde{v}}_{30}(0,t)=0,\ \ \overset{0}{\widetilde{v}}_{30}(l_2,t)=0, \tag{20}$$

$$\overset{0}{\widetilde{p}}_{0}(0,t)=O(1) \tag{21}$$

and initial conditions (ICs)

$$\overset{0}{\widetilde{v}}_{10}(x_2,0)=0,\ \ \overset{0}{\widetilde{v}}_{30}(x_2,0)=0. \tag{22}$$

Problem 2. Let $\overset{0}{X}_1,\ \overset{0}{X}_3\in C([0,l_2])$. Find

$$\begin{aligned}
&\overset{0}{\widetilde{v}}_{10}(\cdot,t),\ \overset{0}{\widetilde{v}}_{30}(\cdot,t)\in C^2(]0,l_2[)\cap C([0,l_2]),\\
&\overset{0}{\widetilde{v}}_{10}(x_2,\cdot),\ \overset{0}{\widetilde{v}}_{30}(x_2,\cdot)\in C(t\geq 0)\cap C^1(t>0),\\
&\overset{0}{\widetilde{v}}_{10}(x_2,t),\ \overset{0}{\widetilde{v}}_{30}(x_2,t)\in C([0,l_2],t\geq 0),
\end{aligned}$$

satisfying equations (16), (17), boundary conditions (19), (20) and initial conditions (22).

Let denote by V and $\overset{0}{X}$ the following vectors

$$V:=\left\{\overset{0}{\widetilde{v}}_{10},\overset{0}{\widetilde{v}}_{30}\right\}\ \ \overset{0}{X}:=\left\{\overset{0}{X}_1,\overset{0}{X}_3\right\}.$$

For V and $\overset{0}{X}$ we can rewrite system (16), (17), BCs and ICs as follows

$$\Big[\mu h V_{,2}\Big]_{,2}+\overset{0}{X}=\rho h\frac{\partial V}{\partial t}, \tag{23}$$

$$V(0,t)=0,\ \ V(l_2,t)=0, \tag{24}$$

$$V(x_2,0)=f(x_2), \tag{25}$$

Let for simplicity $\overset{0}{X}\equiv 0$.
We are looking for V as form as follows

$$V(x_2,t)=X(x_2)T(t).$$

In virtue of (23), (24) for $X(x_2)$ we get

$$T'=-\lambda^2 T,$$

$$(\mu h X')'=-\lambda^2\rho h X, \tag{26}$$

$$X(0)=X(l_2)=0. \tag{27}$$

After two times integration of (26) and using boundary conditions (27) we get the following integral equation

$$X = -\frac{\lambda^2 \rho}{\mu} \int_{x_2}^{l_2} hK(x_2, \xi) X(\xi) d\xi, \tag{28}$$

where

$$K(x_2, \xi) = \begin{cases} -\dfrac{\int_0^{x_2} \frac{d\eta}{h(\eta)} \int_\xi^{l_2} \frac{d\eta}{h(\eta)}}{\int_0^{l_2} \frac{d\eta}{h(\eta)}} = -\dfrac{x_2^{1-\alpha}(l_2^{1-\alpha} - \xi^{1-\alpha})}{h_0 l_2^{1-\alpha}(1-\alpha)}, & x_2 \le \xi \le l_2 \\ \\ -\dfrac{\int_0^{\xi} \frac{d\eta}{h(\eta)} \int_{x_2}^{l_2} \frac{d\eta}{h(\eta)}}{\int_0^{l_2} \frac{d\eta}{h(\eta)}} = -\dfrac{\xi^{1-\alpha}(l_2^{1-\alpha} - x_2^{1-\alpha})}{h_0 l_0 (1-\alpha)}, & 0 \le \xi \le x_2 \end{cases}$$

is a symmetric kernel, which has positive eigenvalues λ_n, whose number is not finite (see [3] Problem 1 pages 91, 92).

The bounded solution of the equation

$$T_n'(t) = -\lambda_n^2 T_n(t)$$

has a form

$$T_n = b_n e^{-\lambda_n^2 t}.$$

Now, we can write a formal solution of the Problem (23)-(25) in the form as follows

$$V(x_2) = \sum_{n=1}^{\infty} X_n(x_2) b_n e^{-\lambda_n^2 t}. \tag{29}$$

In view of initial condition (25), we formally have

$$\sum_{n=1}^{\infty} X_n(x_2) b_n = f(x_2).$$

Therefore,

$$b_n = \int_0^{l_2} X_n(x_2) f(x_2) dx_2.$$

Theorem 2.1 : *If*

$$F := (hf'(x_2))' \text{ is integrable in }]0, l_2[$$

satisfying BCs $F(0) = F(l_2) = 0$, then series on the right hand side of (29) convergence absolutely and uniformly on $[0, l_2]$. Series $V_{,t}$, $V_{,22}$ convergent absolutely and uniformly on any $[a, b] \in]0, l_2[$.

Proof is analogous to the proof of Theorem 2.4 of [3], on pages 93-95.

So, the following Proposition is valid

Proposition 2.2: *Solution of the Problem 2 with homogeneous right hand side has the form (29).*

Remark 1: If $\overset{0}{X}_1 \not\equiv 0$ and $\overset{0}{X}_3 \not\equiv 0$, they can be represented as convergent series

$$\overset{0}{X}_1 = \sum_{n=1}^{\infty} X_n \overset{0}{X}_{1n}, \quad \overset{0}{X}_{1n} := \int_0^{l_2} \overset{0}{X}_1 X_n dx_2,$$

$$\overset{0}{X}_3 = \sum_{n=1}^{\infty} X_n \overset{0}{X}_{3n}, \quad \overset{0}{X}_{3n} := \int_0^{l_2} \overset{0}{X}_3 X_n dx_2,$$

and Problem 2 with inhomogeneous right hand side can be solved analogously.

Problem 3. Let $\overset{0}{X}_2 \in O(x_2^\alpha)$ when $x_2 \to 0+$. Find

$$\overset{0}{\widetilde{p}}_0(\cdot, t) \in C(]0, l_2]) \cap C^1(]0, l_2]), \;\; \overset{2}{p}_0 \in C(]0, l_2], \; l_2 \geq 0)$$

Satisfy equation (18) and BC (21).

Solution of Problem 3 looks like

$$\overset{0}{\widetilde{p}}_0 = (-\frac{\overset{0}{X}_2}{h} - \mu \overset{0}{\widetilde{v}}_{10,2}).$$

3. Conclusions

The solutions of 2D problem then fluid occupied the domain of constant thickens ($2h = const$) and 1D problem then fluid occupied the domain with the thickness as follows

$$2h(x_2) = h_0 x_2^\alpha, \;\; x_2 \in [0, l_2] \;\; h_0, \; l_2 \; \alpha = const > 0, \;\; 0 < \alpha < 1$$

are written in the absolutely and uniformly convergent series.

In the forthcoming paper we investigate dynamical problem of zero approximation of hierarchical models for fluids applying the Laplace transform technique, we reduce the dynamical problem to the elliptic problem which depends on a complex parameter and prove the corresponding uniqueness and existence results.

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