Lecture Notes of TICMI
Vol. 25, 2024, 35–47

On the Construction of General Solutions of Equations of the Plane Theory of Elasticity in the Coupled Theory of Double-Porosity Materials

Guranda Charkseliani[1], Bakur Gulua[1,2*], Roman Janjgava[2]

[1] *Sokhumi State University, 61 Politkovskaya St., Tbilisi 0186, Georgia*
[2] *I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University 11 University St., 0186, Tbilisi, Georgia*

In the present paper, the linear coupled model of elastic double-porosity materials is proposed in which the coupled phenomenon of the concepts of Darcy's extended law and the volume fractions is considered. A two dimensional system of equations of plane deformation is written in the complex form and its general solution is represented by means of three analytic functions of a complex variable and three solutions of Helmholtz equations. The constructed general solution enables one to solve analytically a sufficiently wide class of plane boundary value problems of the elastic equilibrium of the coupled theory of elasticity for double-porous bodies. The specific boundary value problem is solved for a circle.

Keywords: Double-porosity materials; Kolosov-Muskhelishvili formulas; boundary value problems.

AMS Subject Classification: 74F10, 74G05, 74G45.

Introduction

Elastic materials with double-porosity are very common in practice. These include hard tissues of animals and the humans, geological materials such as rocks and soils, manufactured porous materials such as ceramics and pressed powders, and biomaterials such as bone (for details, see Straughan [1], Svanadze [2] and the references therein).

There is significant interest in formulations of mathematical models of multi-porosity solids. In fact, the deformation of porous bodies changes both the volume fraction of pores and the pressure of the fluid in those pores, and vice versa. Recently, on the basis of this mechanical effect, in the papers [3-6], the linear models of elasticity, thermo elasticity and viscoelasticity for single-porosity materials are introduced in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction of pore network is considered.

On the basis of Darcy's extended law the double-porosity models have been proposed by different authors as extensions to the single-porosity model of Biot [7]. The first mathematical model of elastic materials with double porosity was proposed by Wilson and Aifantis [8].

* Corresponding author. Email: bak.gulua@gmail.com

ISSN: 1512-0511 print

By using the concept of volume fraction of pores the theory of elastic materials with single voids is proposed by Nunziato and Cowin [9, 10]. By using the mechanics of materials with voids the theories of elasticity and thermoelasticity for materials with double-porosity structure are presented by Ieşan and Quintanilla [11].

The explicit solutions on some basic boundary value problems in the form of series and in quadratures are given in works [12-22].

In the present paper, the linear mathematical model of double-porosity materials is introduced in which the coupled phenomenon of the concepts of Darcy's law and the volume fractions of two levels of pores (macro- and micropores) is proposed. In the spirit of N.I. Muskhelishvili the governing system of equations of the plane strain is rewritten in the complex form and its general solution is represented by means of three analytic functions of the complex variable and three solutions of Helmholtz equations. The constructed general solution enables us to solve analytically the problems for a circle.

1. Basic equations for materials with double voids of the 3D model

Let $x = (x_1; x_2; x_3)$ be a point of the Euclidean three dimensional space R^3. We assume that the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range $(1; 2; 3)$.

In what follows we consider an isotropic and homogeneous elastic solid with double voids occupying a region of $\Omega \in R^3$. The governing equations of the theory of elastic materials with double voids can be expressed in the following form [5-6]:

• Equations of equilibrium

$$
\begin{aligned}
&t_{ji,j} + \rho_0 f_i = 0, \quad i,j = 1,2,3,\\
&\sigma_{j,j} + \xi + \rho_0 g = 0,\\
&\tau_{j,j} + \zeta + \rho_0 l = 0,
\end{aligned} \tag{1}
$$

where t_{ij} is the symmetric stress tensor, f_i is the body force per unit mass, ρ_0 is the mass density, σ_i and τ_i are the equilibrated stress vectors, ξ and ζ are the intrinsic equilibrated body forces, g is the extrinsic equilibrated body force per unit mass associated to macro pores, l is the extrinsic equilibrated body force per unit mass associated to fissures.

• Constitutive equations

$$
\begin{aligned}
&t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + (b_1\varphi_1 + b_2\varphi_2)\delta_{ij} - (\beta_1 p_1 + \beta_2 p_2)\delta_{ij},\\
&\sigma_i = a_1\varphi_{1,i} + a_3\varphi_{2,i},\\
&\tau_i = a_3\varphi_{1,i} + a_2\varphi_{2,i},\\
&\xi = -b_1 e_{kk} - \alpha_1\varphi_1 - \alpha_3\varphi_2 + m_1 p_1 + m_3 p_2,\\
&\zeta = -b_2 e_{kk} - \alpha_3\varphi_1 - \alpha_2\varphi_2 + m_3 p_1 + m_2 p_2,
\end{aligned} \tag{2}
$$

where λ and μ are the Lamé constants, b_1, b_2, β_1, β_2, a_1, a_2, a_3, α_1, α_2, α_3, m_1, m_2 and m_3 are the constants characterizing the body porosity, δ_{ij} is the Kronecker delta, φ_1 is a changes of volume fraction corresponding to pores, φ_2 is a a changes of volume fraction corresponding to fissures, p_1 and p_2 are the changes of the fluid pressures in macro- and micropore networks, respectively, e_{ij} is the strain tensor and

$$e_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{3}$$

where $u_i,\ i = 1,2,3$ are the components of the displacement vector.

Equations of fluid mass conservation

$$v^{(1)}_{j,j} + \gamma_0(p_1 - p_2) = 0, \quad v^{(2)}_{j,j} - \gamma_0(p_1 - p_2) = 0, \tag{4}$$

where $\mathbf{v}^{(1)} = (v^{(1)}_1, v^{(1)}_2, v^{(1)}_3)$ and $\mathbf{v}^{(2)} = (v^{(2)}_1, v^{(2)}_2, v^{(2)}_3)$ are the fluid flux vectors associated to the macro and micro pore networks, respectively; γ_0 is the internal transport coefficient and corresponds to a fluid transfer rate respecting the intensity of the flow between macro and micro pores, $\gamma_0 \geq 0$.

Darcy's extended law for double-porosity materials

$$\begin{aligned} v^{(1)}_j &= -\tfrac{\kappa_1}{\mu'} p_{1,j} - \tfrac{\kappa_3}{\mu'} p_{2,j} - \rho_1 s_1, \\ v^{(2)}_j &= -\tfrac{\kappa_3}{\mu'} p_{1,j} - \tfrac{\kappa_2}{\mu'} p_{2,j} - \rho_2 s_2, \end{aligned} \tag{5}$$

where μ' is the fluid viscosity, and $\rho_1,\ s_1$ and $\rho_2,\ s_2$ are the density of fluid and the external force (such as gravity) for the pore and fissure networks, respectively.

Substituting Eqs. (2), (3) and (5) into (1) and (4) we obtain the following system of equations of motion in the linear coupled theory of elastic double-porosity materials expressed in terms of the displacement vector $\mathbf{u}$, the changes of the volume fractions φ_1, φ_2 and the pressures p_1, p_2:

$$\begin{aligned} &\mu\tilde{\Delta} u_i + (\lambda + \mu)\partial_i \Theta + b_1 \partial_i \varphi_1 + b_2 \partial_i \varphi_2 - \beta_1 \partial_i p_1 - \beta_2 \partial_i p_2 = 0, \quad j = 1,2,3 \\ &(a_1\tilde{\Delta} - \alpha_1)\varphi_1 + (a_3\tilde{\Delta} - \alpha_3)\varphi_2 - b_1\Theta + m_1 p_1 + m_3 p_2 = 0, \\ &(a_3\tilde{\Delta} - \alpha_3)\varphi_1 + (a_2\tilde{\Delta} - \alpha_2)\varphi_2 - b_2\Theta + m_3 p_1 + m_2 p_2 = 0, \\ &k_1\tilde{\Delta} p_1 + k_3\tilde{\Delta} p_2 - \gamma_0(p_1 - p_2) = 0, \\ &k_3\tilde{\Delta} p_1 + k_2\tilde{\Delta} p_2 + \gamma_0(p_1 - p_2) = 0, \end{aligned}$$

where $\partial_i \equiv \frac{\partial}{\partial x_i}$, $\Theta = \partial_k u_k$, $\Delta \equiv \partial_{11} + \partial_{22} + \partial_{33}$ is the three-dimensional Laplace operator.

The constitutive equations also meet some other conditions, following from physical considerations

$$\begin{aligned} &\mu > 0, \ \ 3\lambda + 2\mu > 0, \ \ a_1 > 0, \ \ a_1 a_2 - a_3^2 > 0, \\ &k_1 > 0, \ \ k_1 k_2 - k_3^2 > 0. \end{aligned} \tag{6}$$

2. Basic (governing) equations of the plane strain

From the basic three-dimensional equations we obtain the basic equations for the case of plane strain. Let Ω be a sufficiently long cylindrical body with generatrix parallel to the Ox_3-axis. Denote by V the crosssection of this cylindrical body, thus $V \subset R^2$. In the case of plane deformation $u_3 = 0$ while the functions u_1, u_2, φ_1, φ_2, p_1 and p_2 do not depend on the coordinate x_3.

As it follows from formulas (2) and (3), in the case of plane strain

$$t_{k3} = t_{3k} = 0, \quad \sigma_3 = 0, \quad \tau_3 = 0, \quad k = 1, 2.$$

Assuming $\Phi_i \equiv 0$ and $\Psi \equiv 0$. Therefore the system of equilibrium equations (1) takes the form

$$\begin{aligned}
&\partial_1 t_{11} + \partial_2 t_{21} = 0,\\
&\partial_1 t_{12} + \partial_2 t_{22} = 0,\\
&\partial_k \sigma_k + \xi = 0,\\
&\partial_k \tau_k + \zeta = 0.
\end{aligned} \tag{7}$$

Now, Relations (2) are rewritten as

$$\begin{aligned}
&t_{11} = \lambda\theta + 2\mu\partial_1 u_1 + b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2,\\
&t_{22} = \lambda\theta + 2\mu\partial_2 u_2 + +b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2,\\
&t_{12} = t_{21} = \mu(\partial_1 u_2 + \partial_2 u_1),\\
&t_{33} = \sigma(t_{11} + t_{22}),\\
&\sigma_k = a_1\partial_k\varphi_1 + a_3\partial_k\varphi_2,\\
&\tau_k = a_3\partial_k\varphi_1 + a_2\partial_k\varphi_2,\\
&\xi = -b_1 e_{kk} - \alpha_1\varphi_1 - \alpha_3\varphi_2 + m_1 p_1 + m_3 p_2,\\
&\zeta = -b_2 e_{kk} - \alpha_3\varphi_1 - \alpha_2\varphi_2 + m_3 p_1 + m_2 p_2,
\end{aligned} \tag{8}$$

where σ is the Poisson ratio, $\theta = \partial_1 u_1 + \partial_2 u_2$.

If relations (8) are substituted into system (7) then we obtain the following system of governing equations of statics with respect to the functions u_1, u_2, φ_1, φ_2, p_1 and p_2

$$\begin{aligned}
&\mu\Delta u_k + (\lambda+\mu)\partial_k\theta + b_1\partial_k\varphi_1 + b_2\partial_k\varphi_2 - \beta_1\partial_k p_1 - \beta_2\partial_k p_2 = 0, \quad k = 1, 2,\\
&(a_1\Delta - \alpha_1)\varphi_1 + (a_3\Delta - \alpha_3)\varphi_2 - b_1\theta + m_1 p_1 + m_3 p_2 = 0,\\
&(a_3\Delta - \alpha_3)\varphi_1 + (a_2\Delta - \alpha_2)\varphi_2 - b_2\theta + m_3 p_1 + m_2 p_2 = 0,\\
&k_1\Delta p_1 + k_3\Delta p_2 - \gamma_0(p_1 - p_2) = 0,\\
&k_3\Delta p_1 + k_2\Delta p_2 + \gamma_0(p_1 - p_2) = 0,
\end{aligned} \tag{9}$$

Note that $\Delta \equiv \partial_{11} + \partial_{22}$ is the two-dimensional Laplace operator.

On the plane Ox_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar{z}} = 0.5(\partial_1 + i\partial_2)$, $\bar{z} = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar{z}}$.

To write system (7) in the complex form, the second equation of this system we multiplied by i and sum up with the first equation

$$
\begin{aligned}
&\partial_z(t_{11} - t_{22} + 2it_{12}) + \partial_{\bar{z}}(t_{11} + t_{22}) = 0,\\
&\partial_z\sigma_+ + \partial_{\bar{z}}\bar{\sigma}_+ + \xi = 0,\\
&\partial_z\tau_+ + \partial_{\bar{z}}\bar{\tau}_+ + \zeta = 0,
\end{aligned}
\tag{10}
$$

where $\sigma_+ = \sigma_1 + i\sigma_2$, $\tau_+ = \tau_1 + i\tau_2$ and formulas (8) we rewrite as follows

$$
\begin{aligned}
&t_{11} - t_{22} + 2it_{12} = 4\mu\partial_{\bar{z}}u_+,\\
&t_{11} + t_{22} = 2(\lambda + \mu)\theta + 2b_1\varphi + 2b_2\varphi - 2\beta_1 p_1 - 2\beta_2 p_2,\\
&\sigma_+ = 2a_1\partial_{\bar{z}}\varphi_1 + 2a_3\partial_{\bar{z}}\varphi_2,\\
&\tau_+ = 2a_3\partial_{\bar{z}}\varphi_1 + 2a_2\partial_{\bar{z}}\varphi_2,\\
&\xi = -b_1\theta - \alpha_1\varphi_1 - \alpha_3\varphi_2 + m_1 p_1 + m_3 p_2,\\
&\zeta = -b_2\theta - \alpha_3\varphi_1 - \alpha_2\varphi_2 + m_3 p_1 + m_2 p_2,
\end{aligned}
\tag{11}
$$

$$
\theta = \partial_z u_+ + \partial_{\bar{z}}\bar{u}_+, \quad u_+ = u_1 + iu_2.
$$

Substituting relations (11) into system (10), we rewrite system (9) in the complex form

$$
\begin{aligned}
&2\mu\partial_{\bar{z}}\partial_z u_+ + (\lambda + \mu)\partial_{\bar{z}}\theta + b_1\partial_{\bar{z}}\varphi_1 + b_2\partial_{\bar{z}}\varphi_2 - \beta_1\partial_{\bar{z}}p_1 - \beta_2\partial_{\bar{z}}p_2 = 0,\\
&(a_1\Delta - \alpha_1)\varphi_1 + (a_3\Delta - \alpha_3)\varphi_2 - b_1\theta + m_1 p_1 + m_3 p_2 = 0,\\
&(a_3\Delta - \alpha_3)\varphi_1 + (a_2\Delta - \alpha_2)\varphi_2 - b_2\theta + m_3 p_1 + m_2 p_2 = 0,\\
&k_1\Delta p_1 + k_3\Delta p_2 - \gamma_0(p_1 - p_2) = 0,\\
&k_3\Delta p_1 + k_2\Delta p_2 + \gamma_0(p_1 - p_2) = 0.
\end{aligned}
\tag{12}
$$

3. Kolosov-Muskhelishvili'analogies formulas for (12) system

Now we construct the analogues to the Kolosov-Muskhelishvili formulas for system (12) [17, 22, 23].

From the fourth and fifth equations of the system (12) we easily obtain the expressions for the pressures p_1 and p_2 [17]

$$
\begin{aligned}
p_1 &= g'(z) + \overline{g'(z)} + (k_2 + k_3)\eta(z,\bar{z}),\\
p_2 &= g'(z) + \overline{g'(z)} - (k_1 + k_3)\eta(z,\bar{z}),
\end{aligned} \tag{13}
$$

where $g'(z)$ is an arbitrary analytic function of a complex variable z,$\eta(z,\bar{z})$ is an arbitrary solution of the Helmholtz equation

$$\triangle\eta - \nu^2\eta = 0,$$

$$\nu^2 = \frac{k_1 + k_2 + k_3^2}{k_1k_2 - k_3^2}\gamma_0.$$

We take the operator $\partial_{\bar{z}}$ out of the brackets in the left-hand part of the first equation of system (12)

$$\partial_{\bar{z}}(2\mu\partial_z u_+ + (\lambda+\mu)\theta + b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2) = 0. \tag{14}$$

Since (13) is a system of Cauchy-Riemann equations, we have

$$2\mu\partial_z u_+ + (\lambda+\mu)\theta + b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2 = Af'(z), \tag{15}$$

where $f(z)$ is an arbitrary analytic function of z and A is an arbitrary constant.

A conjugate equation to (15) has the form

$$2\mu\partial_{\bar{z}}\bar{u}_+ + (\lambda+\mu)\theta + b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2 = A\overline{f'(z)}. \tag{16}$$

Summing up equations (15) and (16) and taking into account that

$$\theta = \partial_z u_+ + \partial_{\bar{z}}\bar{u}_+$$

we obtain

$$\theta = \frac{A}{2(\lambda+2\mu)}(f'(z) + \overline{f'(z)}) - \frac{b_1}{\lambda+2\mu}\varphi_1 - \frac{b_2}{\lambda+2\mu}\varphi_2 + \frac{\beta_1}{\lambda+2\mu}p_1 + \frac{\beta_2}{\lambda+2\mu}p_2. \tag{17}$$

Substituting formula (17) into the second and third equations of system (12), we have

$$
\begin{aligned}
&\left(a_1\Delta - \alpha_1 + \frac{b_1^2}{\lambda+2\mu}\right)\varphi_1 + \left(a_3\Delta - \alpha_3 + \frac{b_1b_2}{\lambda+2\mu}\right)\varphi_2\\
&= \frac{Ab_1}{2(\lambda+2\mu)}(f'(z) + \overline{f'(z)}) + \left(\frac{b_1\beta_1}{\lambda+2\mu} - m_1\right)p_1 + \left(\frac{b_1\beta_2}{\lambda+2\mu} - m_3\right)p_2,\\
&\left(a_3\Delta - \alpha_3 + \frac{b_1b_2}{\lambda+2\mu}\right)\varphi_1 + \left(a_2\Delta - \alpha_2 + \frac{b_2^2}{\lambda+2\mu}\right)\varphi_2\\
&= \frac{Ab_2}{2(\lambda+2\mu)}(f'(z) + \overline{f'(z)}) + \left(\frac{b_2\beta_1}{\lambda+2\mu} - m_3\right)p_1 + \left(\frac{b_2\beta_2}{\lambda+2\mu} - m_2\right)p_2.
\end{aligned} \tag{18}
$$

(18) system rewrite in matrix form

$$\Delta\Psi - C\Psi = DF + TG + TH, \tag{19}$$

where

$$C = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \alpha_1 - \dfrac{b_1^2}{\lambda+2\mu} & \alpha_3 - \dfrac{b_1 b_2}{\lambda+2\mu} \\ \alpha_3 - \dfrac{b_1 b_2}{\lambda+2\mu} & \alpha_2 - \dfrac{b_2^2}{\lambda+2\mu} \end{pmatrix},$$

$$D = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \dfrac{Ab_1}{2(\lambda+2\mu)} & 0 \\ 0 & \dfrac{Ab_2}{2(\lambda+2\mu)} \end{pmatrix},$$

$$T = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \dfrac{b_1\beta_1}{\lambda+2\mu} - m_1 & \dfrac{b_1\beta_2}{\lambda+2\mu} - m_3 \\ \dfrac{b_2\beta_1}{\lambda+2\mu} - m_3 & \dfrac{b_2\beta_2}{\lambda+2\mu} - m_2 \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad F = \begin{pmatrix} f'(z) + \overline{f'(z)} \\ f'(z) + \overline{f'(z)} \end{pmatrix},$$

$$G = \begin{pmatrix} g'(z) + \overline{g'(z)}) \\ g'(z) + \overline{g'(z)}) \end{pmatrix},$$

$$H = \begin{pmatrix} (k_2+k_3)\eta(z,\bar z) \\ -(k_1+k_3)\eta(z,\bar z) \end{pmatrix}.$$

The general solutions of system (19) we may write in the form

$$\begin{aligned} \varphi_1 &= l_{11}\chi_1(z,\bar z) + l_{12}\chi_2(z,\bar z) - A\bar e_1(f'(z) + \overline{f'(z)}) \\ &\quad - e_3(g'(z) + \overline{g'(z)}) - e_5\eta(z,\bar z), \\ \varphi_2 &= l_{21}\chi_1(z,\bar z) + l_{22}\chi_2(z,\bar z) - A\bar e_2(f'(z) + \overline{f'(z)}) \\ &\quad - e_4(g'(z) + \overline{g'(z)}) - e_6\eta(z,\bar z), \end{aligned} \tag{20}$$

where $\chi_1(z,\bar z)$ and $\chi_2(z,\bar z)$ are general solutions of the Helmholtz equations

$$\Delta\chi(z,\bar z) - \kappa_1\chi(z,\bar z) = 0, \quad \Delta\chi(z,\bar z) - \kappa_2\chi(z,\bar z) = 0.$$

where κ_α are eigenvalues and $(l_{11},\ l_{21})$, $(l_{12},\ l_{22})$ are eigenvectors of the matrix C and from (6), they are positive numbers,

$$\bar{e}_1 = \frac{b_1\alpha_2 - b_2\alpha_3}{2((\alpha_1\alpha_2 - \alpha_3^2)(\lambda + 2\mu) - \alpha_1 b_2^2 - \alpha_2 b_1^2 + 2\alpha_3 b_1 b_2)},$$

$$\bar{e}_2 = \frac{b_1\alpha_1 - b_1\alpha_3}{2((\alpha_1\alpha_2 - \alpha_3^2)(\lambda + 2\mu) - \alpha_1 b_2^2 - \alpha_2 b_1^2 + 2\alpha_3 b_1 b_2)},$$

$$e_3 = T^*_{11} + T^*_{12}, \quad e_4 = T^*_{12} + T^*_{22},$$

$$e_5 = X_{11}(\kappa_2 + \kappa_3) - X_{12}(\kappa_1 + \kappa_3),$$

$$e_6 = X_{21}(\kappa_2 + \kappa_3) - X_{22}(\kappa_1 + \kappa_3),$$

where T^*_{11}, T^*_{12} and T^*_{22} are elements of matrix $T^* = C^{-1}T$, X_{11} , X_{12}, X_{21} and X_{22} are elements of the matrix $X = (\zeta^2 I - C)^{-1}T$.

Substituting formulas (17) and (20) into equation (16), we obtain

$$2\mu\partial_z u_+ = \frac{\lambda + 3\mu + 2\mu(b_1e_1 + b_2e_2)}{2(\lambda + 2\mu)} Af'(z) - \frac{\lambda + \mu - 2\mu(b_1e_1 + b_2e_2)}{2(\lambda + 2\mu)}\overline{f'(z)}$$

$$+\frac{\mu(b_1e_3 + b_2e_4 + \beta_1 + \beta_2)}{\lambda + 2\mu}(g'(z) + \overline{g'(z)}) - \frac{\mu(b_1l_{11} + b_2l_{21})}{\lambda + 2\mu}\chi_1(z, \bar{z})$$

$$-\frac{\mu(b_1l_{12} + b_2l_{22})}{\lambda + 2\mu}\chi_2(z, \bar{z}) + \frac{\mu(b_1e_5 + b_2e_6 + \beta_1(\kappa_2 + \kappa_3) - \beta_2(\kappa_1 + \kappa_3))}{\lambda + 2\mu}\eta(z, \bar{z}).$$

Now, let $A := \dfrac{2(\lambda + 2\mu)}{\lambda + \mu - 2\mu(b_1e_1 + b_2e_2)}$ then we get

$$2\mu u_+ = \varkappa f(z) - z\overline{f'(z)} - \overline{h(z)} + q_1(g(z) + z\overline{g'(z)}),$$

$$-q_2\partial_{\bar{z}}\chi_1(z, \bar{z}) - q_3\partial_{\bar{z}}\chi_2(z, \bar{z}) + q_4\partial_{\bar{z}}\eta(z, \bar{z}),$$

where

$$\varkappa = \frac{\lambda + 3\mu + 2\mu(b_1e_1 + b_2e_2)}{\lambda + \mu - 2\mu(b_1e_1 + b_2e_2)}, \quad q_1 = \frac{\mu(b_1e_3 + b_2e_4 + \beta_1 + \beta_2)}{\lambda + 2\mu}$$

$$q_2 = \frac{4\mu(b_1l_{11} + b_2l_{21})}{\kappa_1(\lambda + 2\mu)} \quad q_3 = \frac{4\mu(b_1l_{12} + b_2l_{22})}{\kappa_2(\lambda + 2\mu)},$$

$$q_4 = \frac{4\mu(b_1e_5 + b_2e_6 + \beta_1(\kappa_2 + \kappa_3) - \beta_2(\kappa_1 + \kappa_3))}{\nu^2(\lambda + 2\mu)},$$

$h(z)$ is an arbitrary analytic function of z.
Thus, we have proved

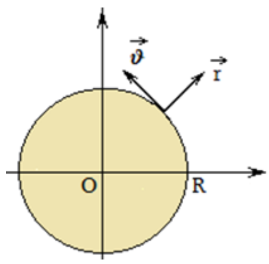
Theorem 3.1 : *The general solution of the system (12) is represented as follows:*

$$2\mu u_+ = \varkappa f(z) - z\overline{f'(z)} - \overline{h(z)} + q_1(g(z) + z\overline{g'(z)}) - q_2\partial_{\bar z}\chi_1(z,\bar z) - q_3\partial_{\bar z}\chi_2(z,\bar z)$$
$$+q_4\partial_{\bar z}\eta(z,\bar z),$$
$$\varphi_1 = l_{11}\chi_1(z,\bar z) + l_{12}\chi_2(z,\bar z) - e_1(f'(z) + \overline{f'(z)}) - e_3(g'(z) + \overline{g'(z)}) - e_5\eta(z,\bar z),$$
$$\varphi_2 = l_{21}\chi_1(z,\bar z) + l_{22}\chi_2(z,\bar z) - e_2(f'(z) + \overline{f'(z)}) - e_4(g'(z) + \overline{g'(z)}) - E_6\eta(z,\bar z),$$
$$p_1 = g'(z) + \overline{g'(z)} + (k_2 + k_3)\eta(z,\bar z),$$
$$p_2 = g'(z) + \overline{g'(z)} - (k_1 + k_3)\eta(z,\bar z),$$

where $e_1 = A\bar e_1$, $e_2 = A\bar e_2$.

4. The Dirichlet problem for a circle

Let the elastic circle bounded by the circumference of radius R. The origin of coordinates is at the center of the circle.



On the circumference we consider the following boundary value problem

$$\begin{aligned}
&2\mu u_+ = B, \quad \phi = C, \quad \text{on } |z| = R,\\
&\varphi_1 = M', \quad \text{on } |z| = R,\\
&\varphi_2 = M'', \quad \text{on } |z| = R, \qquad (21)\\
&p_1 = T', \quad \text{on } |z| = R,\\
&p_1 = T'', \quad \text{on } |z| = R,
\end{aligned}$$

where B, M', M'', T' and T'' are sufficiently smooth functions.

The analytic functions $f(z)$, $h(z)$, $g(z)$ and the metaharmonic functions $\chi_1(z,\bar{z})$, $\chi_2(z,\bar{z})$, $\eta(z,\bar{z})$ are represented as the series

$$f(z)=\sum_{n=1}^{\infty}a_n z^n,\quad h(z)=\sum_{n=0}^{\infty}b_n z^n,\quad g(z)=\sum_{n=1}^{\infty}c_n z^n, \tag{22}$$

$$\chi_1(z,\bar{z})=\sum_{-\infty}^{+\infty}\alpha_n' I_n(\sqrt{\kappa_1}r)e^{in\vartheta},\ \ \chi_2(z,\bar{z})=\sum_{-\infty}^{+\infty}\beta_n' I_n(\sqrt{\kappa_2}r)e^{in\vartheta}, \tag{23}$$

$$\eta(z,\bar{z})=\sum_{-\infty}^{+\infty}\gamma_n' I_n(\nu r)e^{in\vartheta}, \tag{24}$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind of n-th order. After substituting into the boundary conditions (21-24) we have

$$\begin{aligned}
&\varkappa\sum_{n=1}^{\infty}R^n a_n e^{in\vartheta}-\sum_{n=1}^{\infty}nR^n\bar{a}_n e^{-i(n-2)\vartheta}-\sum_{n=0}^{\infty}R^n\bar{b}_n e^{-in\vartheta}+q_1\sum_{n=1}^{\infty}R^n c_n e^{in\vartheta}\\
&+q_1\sum_{n=1}^{\infty}nR^n\bar{c}_n e^{-i(n-2)\vartheta}-\frac{q_2\sqrt{\kappa_1}}{2}\sum_{-\infty}^{\infty}\alpha_n' I_{n+1}(\sqrt{\kappa_1}R)e^{i(n+1)\vartheta}\\
&-\frac{q_3\sqrt{\kappa_2}}{2}\sum_{-\infty}^{\infty}\beta_n' I_{n+1}(\sqrt{\kappa_2}R)e^{i(n+1)\vartheta}+\frac{q_4\nu}{2}\sum_{-\infty}^{\infty}\gamma_n' I_{n+1}(\nu R)e^{i(n+1)\vartheta}=B,
\end{aligned} \tag{25}$$

$$\begin{aligned}
&l_{11}\sum_{-\infty}^{\infty}\alpha_n' I_n(\sqrt{\kappa_1}R)e^{in\vartheta}+l_{12}\sum_{-\infty}^{\infty}\beta_n' I_n(\sqrt{\kappa_2}R)e^{in\vartheta}\\
&-e_1\sum_{n=1}^{\infty}nR^{n-1}\left(a_n e^{i(n-1)\vartheta}+\bar{a}_n e^{-i(n-1)\vartheta}\right)\\
&-e_3\sum_{n=1}^{\infty}nR^{n-1}\left(c_n e^{i(n-1)\vartheta}+\bar{c}_n e^{-i(n-1)\vartheta}\right)-e_5\sum_{-\infty}^{\infty}\gamma_n' I_n(\nu R)e^{in\vartheta}=M',
\end{aligned} \tag{26}$$

$$\begin{aligned}
&l_{21}\sum_{-\infty}^{\infty}\alpha_n' I_n(\sqrt{\kappa_1}R)e^{in\vartheta}+l_{22}\sum_{-\infty}^{\infty}\beta_n' I_n(\sqrt{\kappa_2}R)e^{in\vartheta}\\
&-e_2\sum_{n=1}^{\infty}nR^{n-1}\left(a_n e^{i(n-1)\vartheta}+\bar{a}_n e^{-i(n-1)\vartheta}\right)\\
&-e_4\sum_{n=1}^{\infty}nR^{n-1}\left(c_n e^{i(n-1)\vartheta}+\bar{c}_n e^{-i(n-1)\vartheta}\right)-e_6\sum_{-\infty}^{\infty}\gamma_n' I_n(\nu R)e^{in\vartheta}=M'',
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} nR^{n-1}\left(c_n e^{i(n-1)\vartheta}+\bar{c}_n e^{-i(n-1)\vartheta}\right)+(\kappa_2+\kappa_3)\sum_{-\infty}^{\infty}\gamma'_n I_n(\nu R)e^{in\vartheta}=T',\\
&\sum_{n=1}^{\infty} nR^{n-1}\left(c_n e^{i(n-1)\vartheta}+\bar{c}_n e^{-i(n-1)\vartheta}\right)-(\kappa_1+\kappa_3)\sum_{-\infty}^{\infty}\gamma'_n I_n(\nu R)e^{in\vartheta}=T''.
\end{aligned} \tag{28}$$

Expand the function B, M', M'', T' and T'' given on $r=R$, in a complex Fourier series

$$B=\sum_{-\infty}^{\infty}A_n e^{in\alpha},\quad M'=\sum_{-\infty}^{\infty}M'_n e^{in\alpha},\quad M''=\sum_{-\infty}^{\infty}M''_n e^{in\alpha},$$

$$T'=\sum_{-\infty}^{\infty}T'_n e^{in\alpha},\quad T''=\sum_{-\infty}^{\infty}T''_n e^{in\alpha}.$$

Comparing in (25-28) the coefficients of $e^{in\vartheta}$

$$\begin{aligned}
&(\varkappa a_1-\bar{a}_1)R+(c_1+\bar{c}_1)q_1R-\frac{q_2\sqrt{\kappa_1}}{2}I_1(\sqrt{\kappa_1}R)\alpha'_0-\frac{q_3\sqrt{\kappa_2}}{2}I_1(\sqrt{\kappa_3}R)\beta'_0\\
&\qquad+\frac{q_4\nu}{2}I_1(\nu R)\gamma'_0=A'_1,\\
&\varkappa R^n a_n+q_1R^n c_n-\frac{q_2\sqrt{\kappa_1}}{2}I_n(\sqrt{\kappa_1}R)\alpha'_{n-1}-\frac{q_3\sqrt{\kappa_2}}{2}I_n(\sqrt{\kappa_2}R)\beta'_{n-1}\\
&\qquad+\frac{q_4\nu}{2}I_n(\nu R)\gamma'_{n-1}=A'_n,\quad n>1,\\
&-(n+2)R^{n+2}\bar{a}_{n+2}-R^n\bar{b}_n+q_1(n+2)R^{n+2}\bar{c}_{n+2}-\frac{q_2\sqrt{\kappa_1}}{2}I_n(\sqrt{\kappa_1}R)\alpha'_{-n-1}\\
&\qquad-\frac{q_3\sqrt{\kappa_2}}{2}I_n(\sqrt{\kappa_2}R)\beta'_{-n-1}+\frac{q_4\nu}{2}I_n(\nu R)\gamma'_{-n-1}=A'_{-n},\quad n\ge 0;
\end{aligned} \tag{29}$$

$$\begin{aligned}
&l_{11}I_0(\sqrt{\kappa_1}R)\alpha'_0+l_{12}I_0(\sqrt{\kappa_2}R)\beta'_0-(a_1+\bar{a}_1)e_1-(c_1+\bar{c}_1)e_3\\
&\qquad-e_5I_0(\nu R)\gamma'_0=M'_0,\\
&l_{21}I_0(\sqrt{\kappa_1}R)\alpha'_0+l_{22}I_0(\sqrt{\kappa_2}R)\beta'_0-(a_1+\bar{a}_1)e_2-(c_1+\bar{c}_1)e_4\\
&\qquad-e_6I_0(\nu R)\gamma'_0=M''_0;
\end{aligned} \tag{30}$$

$$\begin{aligned}
&l_{11}I_0(\sqrt{\kappa_1}R)\alpha'_n+l_{12}I_n(\sqrt{\kappa_2}R)\beta'_n-e_1(n+1)R^n a_{n+1}-e_3(n+1)R^n c_{n+1}\\
&\qquad-e_5I_n(\nu R)\gamma'_n=M'_n,\quad n>0,\\
&l_{21}I_n(\sqrt{\kappa_1}R)\alpha'_n+l_{22}I_n(\sqrt{\kappa_2}R)\beta'_n-e_2(n+1)R^n a_{n+1}-e_4(n+1)R^n c_{n+1}\\
&\qquad-e_6I_n(\nu R)\gamma'_n=M''_n;
\end{aligned} \tag{31}$$

$$\begin{aligned}
&nR^{n-1}c_n+(\kappa_2+\kappa_3)I_{n-1}(\nu R)\gamma'_{n-1}=T'_{n-1},\quad n>0\\
&nR^{n-1}c_n-(\kappa_1+\kappa_3)I_{n-1}(\nu R)\gamma'_{n-1}=T''_{n-1},\quad n>0.
\end{aligned} \tag{32}$$

In order for the problem to have a solution, the following condition must be fulfilled

$$\frac{T_0'}{\kappa_2+\kappa_3}=-\frac{T_0''}{\kappa_1+\kappa_3}.$$

From Eqs. (32) we determine the coefficients c_n and γ_n'

$$\gamma_0'=\frac{T_0'}{\kappa_2+\kappa_3},\ \ \gamma_{n-1}'=\frac{T_{n-1}''-T_{n-1}'}{(\kappa_1+\kappa_2+2\kappa_3)I_{n-1}(\nu R)},\ \ n>1,$$

$$c_n=\frac{(\kappa_1+\kappa_3)T_{n-1}'+(\kappa_2+\kappa_3)T_{n-1}''}{nR^{n-1}(\kappa_1+\kappa_2+2\kappa_3)I_{n-1}(\nu R)},\ \ n>0.$$

From (29)-(31) we can find all the coefficients a_n, b_n, α_n' and β_n'.

The procedure of solving a boundary value problem remains the same when stresses, the equilibrated stress vectors, and change in volume fraction on the domain boundary are given arbitrarily, but the condition that the principal vector and the principal moment of external forces are equal to zero is fulfilled.

References

[1] B. Straughan. *Mathematical Aspects of Multi-Porosity Continua. Advances in Mechanics and Mathematics*, vol. 38. Cham, Switzerland: Springer Int. Publ. AG, 2017

[2] M. Svanadze. *Potential Method in Mathematical Theories of Multi-Porosity Media*, Interdisciplinary Applied Mathematics, vol. 51. Cham, Switzerland: Springer Nature Switzerland AG, 2019

[3] M. Svanadze. *Boundary integral equations method in the coupled theory of thermoelasticity for porous materials*, Proceedings of ASME, IMECE 2019, Vol. 9: Mechanics of Solids, Structures, and Fluids, V009T11A033, November (2019), 11-14

[4] M. M. Svanadze. *Potential method in the coupled theory of viscoelasticity of porous materials*, J. Elast., **144**, 2 (2021), 119-140

[5] M. Svanadze. *Potential Method in the coupled theory of elastic double-porosity materials*, Acta Mech., **232**, 6 (2021), 2307-2329

[6] M. Svanadze. *On the coupled theory of thermoelastic double-porosity materials*, Journal of Thermal Stresses, **45**, 7 (2022), 576-596

[7] M.A. Biot. *General theory of three-dimensional consolidation*, J. Appl. Phys. **12** (1941), 155-164

[8] R.K. Wilson, E.C. Aifantis. *On the theory of consolidation with double porosity*, Int. J. Eng. Sci. **20** (1982), 1009-1035

[9] J. W. Nunziato, S. C. Cowin. *A nonlinear theory of elastic materials with voids*, Arch. Rational Mech. Anal., **72**, 2 (1979), 175-201

[10] S. C. Cowin, J. W. Nunziato. *Linear elastic materials with voids*, J. Elasticity, **13**, 2 (1983), 125-147.

[11] D. Ieş, R. Quintanilla. *On a theory of thermoelastic materials with a double porosity structure*, J. Therm. Stress. **37** (2014), 1017-1036

[12] L. Bitsadze. *Explicit solution of the Dirichlet boundary value problem of elasticity for porous infinite strip*, Zeit. Angew. Math. Phys., **71** (2020), article number 145

[13] L. Bitsadze. *Explicit solutions of quasi-static problems in the coupled theory of poroelasticity*, Continuum Mech. Thermodyn., **33**, 6 (2021), 2481-2492

[14] L. Bitsadze, I. Tsagareli. *The solution of the Dirichlet BVP in the fully coupled theory of elasticity for spherical layer with double porosity*, MeccanicaThis, **51**, 6 (2016), 1457-1463

[15] I. Tsagareli. *Dynamic problems for elastic bodies with double voids*, ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik, **102**, 4 (2022), e202000335

[16] I. Tsagareli. *Solution of the problems of thermoelasticity for a circle with double porosity*, Journal of Thermal Stresses, **46**, 2 (2023), 133-139

[17] R. Janjgava. *Elastic equilibrium of porous Cosserat media with double porosity*, Advances in Mathematical Physics, (2016), 4792148

[18] B. Gulua, R. Janjgava. *Boundary Value Problems of the Plane Theory of Elasticity for Materials with Voids*, Springer Proceedings in Mathematics and StatisticsThis, **334** (2020), 227-236

[19] R. Janjgava, B. Gulua, S. Tsotniashvili. *Some boundary value problems for a micropolar porous elastic body*, Archives of MechanicsThis, **72**, 6 (2020), 485-509

[20] B. Gulua, T. Kasrashvili. *Some basic problems of the plane theory of elasticity for materials with voids*, Seminar of I. Vekua Institute of Applied Mathematics REPORTS, **46** (2020), 27-36

[21] B. Gulua. *Basic boundary value problems for circular ring with voids*, Transactions of A. Razmadze Mathematical Institute, **175** 3 (2021), 437-441

[22] B. Gulua. *Analogues the Kolosov-Muskhelishvili Formulas for Isotropic Materials with Double Voids*, Analysis, Applications, and Computations. ISAAC 2021. Trends in Mathematics, (2023), 373-382

[23] N.I. Muskhelishvili. *Some Basic Problems of the Mathematical Theory of Elasticity.* Fundamental equations, plane theory of elasticity, torsion and bending. Fifth revised and enlarged edition (Russian), Nauka, Moscow, 1966