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Fractional-Type Laplace Transforms

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Since the Laplace transform plays a central role in the solution of differential equations, it seems natural to extend it in the field of fractional calculus, since many applications of this topic have been proposed, and are becoming more and more important. In this paper we extend the classical Laplace Transform by replacing the usual kernel with a suitable one, both in the classical and Laguerre-type case, obtained by constructing the reciprocal of some exponential-type functions with respect to an appropriate differential operator. Some examples are shown, derived using the computer algebra system Mathematica©.

Keywords: Fractional exponential functions, Laguerre-type exponentials, generalized Blissard problem, Laplace transform theory.

AMS Subject Classification: 11B73, 33B99, 44A10.

1. Introduction

In recent years several articles have been devoted to the study of generalized forms of special polynomials and numbers (see e.g. [1–5]. At the same time, the study of the fractional derivative and its applications has had an extraordinary expansion, as can be seen in the monographs [6, 7]. These two fields of research can be combined by studying special functions of index fractional, as is done, for example in [8, 9], where even extensions to the Laguerre-type functions have been considered.

We recall that as the exponential function e^{ax} is an eigenfunction of the derivative operator, since

$$De^{ax} = ae^{ax} \tag{1}$$

(where $D := d/dx$, and a is real or complex number), likewise the *Laguerre-type exponential*

$$e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} \tag{2}$$

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is an eigenfunction of the so called *Laguerre derivative*,

$$\hat{D}_L := DxD = D + x\,D^2\,, \tag{3}$$

since

$$\hat{D}_L\; e_1(ax) = ae_1(ax)\,. \tag{4}$$

In preceding articles, we have shown the role of the Laguerre derivative in the framework of the *monomiality principle* [10, 11], and its application to the multidimensional Hermite (Hermite-Kampé de Fériet or Gould-Hopper polynomials) [12–14] or Laguerre polynomials [10, 15].

The above result has been extended as follows [16, 17].

Considering the differential operator, containind $n+1$ derivatives

$$\begin{aligned}\hat{D}_{(n-1)L} &:= Dx\cdots DxDxD = D\left(xD + x^2D^2 + \cdots + x^{n-1}D^{n-1}\right)\\ &= S(n,1)D + S(n,2)xD^2 + \cdots + S(n,n)x^{n-1}D^n,\end{aligned} \tag{5}$$

and the function:

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}. \tag{6}$$

We have proven in [16] that the function $e_n(ax)$ is an eigenfunction of the operator $\hat{D}_{nL}$, that is

$$\hat{D}_{nL}\; e_n(ax) = ae_n(ax). \tag{7}$$

Remark 1: For completeness, we recall that the operators $D_L = DxD$ and its iterates as $D_{nL} = DxDxDx\cdots DxD$ can be considered as particular cases of the hyper-Bessel differential operators when $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 1$ (the special case considered in operational calculus by Ditkin and Prudnikov [18]). In general, the *Bessel-type differential operators of arbitrary order* n were introduced by Dimovski, in 1966 [19] and later called by Kiryakova *hyper-Bessel operators*, because are closely related to their eigenfunctions, called hyper-Bessel by Delerue [20], in 1953. These operators were studied in 1994 by Kiryakova in her book [7], Ch. 3.

The purpose of this article is to introduce some generalized types of the Laplace transform.

We use expansions of the type $g(x) = \sum_{k=0}^{\infty} a_k x^k$, converging in all the plane, satisfying the eigenvalue property $\hat{D}_x g(ax) = \lambda(a)g(x)$, with respect to a given differential operator $\hat{D}_x$, similar to that of the exponential function. After constructing the reciprocal $[g(x)]^{-1}$ of the considered expansion, we substitute this reciprocal function in the place of $\exp(-xs)$ in the Laplace transormation. So we get the new Laplace-type transform

$$\mathcal{L}_g(f) := \int_0^{+\infty} [g(xs)]^{-1} f(x)dx = \mathcal{F}_g(s)\,.$$

We use the fractional exponential function $\mathrm{Exp}_\alpha(x)$, defined for every α, with $\alpha > 0$, by the equation

$$\mathrm{Exp}_\alpha(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \dots ,$$

as well as the generalized Laguerre-type fractional exponential function

$$e_{a,1}(x) := \sum_{k=0}^{\infty} \frac{a_k\, x^{k\alpha}}{[\Gamma(k\alpha+1)]^2} ,$$

depending on a given sequence of coefficients denoted by the umbral symbol $a := \{a_k\} = \{a_0, a_1, a_2, \dots\}$, which is an eigenfunction of the operator $D_L^\alpha := D_x^\alpha x^\alpha D_x^\alpha$, since

$$D_x^\alpha\, x^\alpha D_x^\alpha e_{a,1}(tx) = t^\alpha e_{a,1}(tx) .$$

Remark 2: Recalling the Mittag-Leffler function [21] $\mathrm{E}_\alpha(x)$, and substituting x with x^α, it results

$$\mathrm{E}_\alpha(x^\alpha) = \mathrm{Exp}_\alpha(x) , \quad \text{and} \quad D_x^\alpha \mathrm{E}_\alpha(x^\alpha) = \mathrm{E}_\alpha(x^\alpha) ,$$

so that the fractional exponential function can be reduced to the Mittag-Leffler function.

Another possibility is to consider the hyperbolic function

$$\cosh(x) := \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} , \tag{8}$$

or, in general, the fractional Laguerre-type case, which includes all possibilities

$$\cosh_{n,\alpha}(x) := \sum_{k=0}^{\infty} \frac{x^{2k\alpha}}{\Gamma(2k\alpha+1)^{n+1}} . \tag{9}$$

The basic function (8) satisfies

$$D^2 \cosh(ax) = a^2 \cosh(ax) , \tag{10}$$

and the function (9), assuming $n = 1$, is an eigenfunction of the fractional operator $D_x^\alpha\, x^\alpha D_x^\alpha$, since

$$D_x^\alpha\, x^\alpha D_x^\alpha \cosh_{1,\alpha}(ax) := a^2 \cosh_{1,\alpha}(ax) . \tag{11}$$

The general case is more involved, but the technique is the same.

We exploit the reciprocal of these unusual exponentials, obtained using an extension of the Blissard problem, to construct generalized forms of the Laplace transform.

In all cases a generalized type of Laplace transform can be defined, and some numerical check is performed using the computer algebra program Mathematica©.

2. A general result

In literature there exists the following general result [22].

Consider the sequences $a := \{a_k\} = (a_0, a_1, a_2, a_3, \dots)$, and $b := \{b_k\} = (b_0, b_1, b_2, b_3, \dots)$, Using the umbral formalism (that is, letting $a_k \equiv a^k$ and $b_k \equiv b^k$), the solution of the equation

$$\frac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{a^n t^{n\alpha}}{\Gamma(n\alpha+1)}} = \sum_{n=0}^{\infty} \frac{b^n t^{n\alpha}}{\Gamma(n\alpha+1)}\,, \quad \text{i.e.}\quad \mathrm{Exp}_\alpha(a\,t)\,\mathrm{Exp}_\alpha(b\,t) = 1\,, \tag{12}$$

according to the Faà di Bruno formula, is given by

$$\begin{aligned} b_n = {} & \frac{\Gamma(n\alpha+1)}{n!} \sum_{k=0}^{n} (-1)^k\, k!\, a_0^{-(k+1)} \cdot \\ & \cdot B_{n,k}\left(\frac{1!a_1}{\Gamma(\alpha+1)}, \frac{2!a_2}{\Gamma(2\alpha+1)}, \dots, \frac{(n-k+1)!a_{n-k+1}}{\Gamma((n-k+1)\alpha+1)}\right), \quad (\forall\, n \geq 0), \end{aligned} \tag{13}$$

where $B_{n,k}$ are partial Bell polynomials [22, 23].

3. The reciprocal of the fractional exponential operator

We consider the generalized fractional exponential operator

$$\mathrm{Exp}_\alpha(a,t) = a_0 + a_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + a_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots + a_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \dots\,, \tag{14}$$

where the symbol a denotes the sequence of coefficients, that is $\{a\} := \{a_0, a_1, a_2, \dots\}$.

The equation

$$\frac{1}{\mathrm{Exp}_\alpha(a,t)} = b_0 + b_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + b_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots + b_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \dots$$

in terms of the unknown sequence $\{b_n\}$ can be solved by using the Blissard problem and Bell's polynomials.

In the particular case of the reciprocal of the $\mathrm{Exp}_\alpha(t)$ function we must assume $\{a\} \equiv \{1, 1, 1, \dots\}$ and therefore, recalling that $B_{n,h}(1, 1, \dots, 1) \equiv S_{n,h}$, that is the Stirling numbers of the second kind, it results

$$\begin{aligned} [\mathrm{Exp}_\alpha(t)]^{-1} &= 1 + \sum_{n=1}^{\infty} \sum_{h=1}^{n} (-1)^h\, \Gamma(h\alpha+1)\, B_{n,h}(1, 1, \dots, 1)\, t^{n\alpha} \\ &= \sum_{n=0}^{\infty} \sum_{h=0}^{n} (-1)^h\, \Gamma(h\alpha+1)\, S_{n,h}\, \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\,, \end{aligned} \tag{15}$$

where we have put $S_{0,0} := 1$.

3.1. *A fractional-type Laplace transform*

Using the above definition for the reciprocal of the fractional exponential, we can introduce a fractional version (of order α, $\alpha > 0$) of the Laplace Transform, by setting

$$\begin{gathered}\mathcal{L}_\alpha(f) := \int_0^\infty f(t)\,[\mathrm{Exp}_\alpha(st)]^{-1}\,dt = \mathcal{F}_\alpha(s) \\ = \int_0^\infty f(t)\left(\sum_{n=0}^{\infty}\sum_{h=0}^{n}(-1)^h\,\Gamma(h\alpha+1)\,S_{n,h}\,\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)dt.\end{gathered} \tag{16}$$

In what follows, we make a comparison among the classical Laplace Transform of assigned functions and the fractional order Laplace transforms of order $\alpha = 1/2$ and $\alpha = 3/2$.

As it is shown in the obtained results, in all cases the graphs of the modulus and argument of the ordinary Laplace Transform lies between the corresponding graphs of the two considered fractional order Laplace transforms. This provides a graphical evidence of the monotonicity property satisfied by the fractional order Laplace transforms.

4. Numerical Examples

4.1. *Example 1*

Consider the fractional Laplace Transforms $\mathcal{F}_{1/2}$, $\mathcal{F}_{3/2}$ of the Bessel function $J_0(2\sqrt{t})$, compared with the classical LT $\mathcal{F} = \mathcal{F}_1$ of the same function.

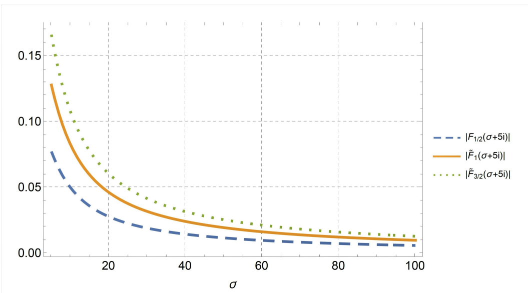

Figure 1. Comparing the fractional LTs $\mathcal{F}_{1/2}, \mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the modulus, assuming $s = \sigma + 5i$

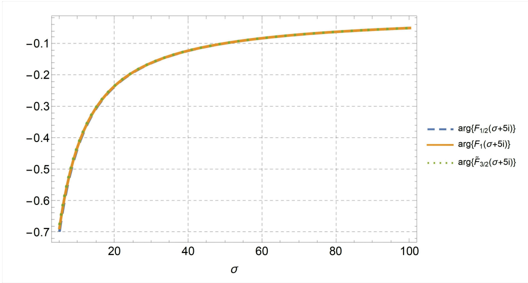

Figure 2. Comparing the fractional LTs $\mathcal{F}_{1/2}$, $\mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the argument, assuming $s = \sigma + 5i$

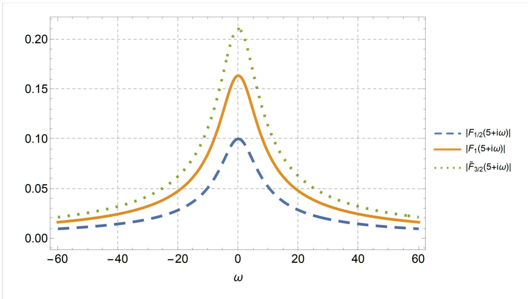

Figure 3. Comparing the fractional LTs $\mathcal{F}_{1/2}$, $\mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the modulus, assuming $s = 5 + i\,\omega$

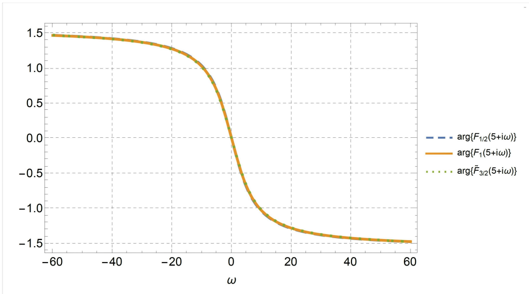

Figure 4. Comparing the fractional LTs $\mathcal{F}_{1/2}$, $\mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the argument, assuming $s = 5 + i\,\omega$

4.2. *Example 2*

Consider the fractional Laplace Transforms $\mathcal{F}_{1/2}$, $\mathcal{F}_{3/2}$ of the function $\exp(-t^2)$, compared with the classical LT $F = \mathcal{F}_1$ of the same function.

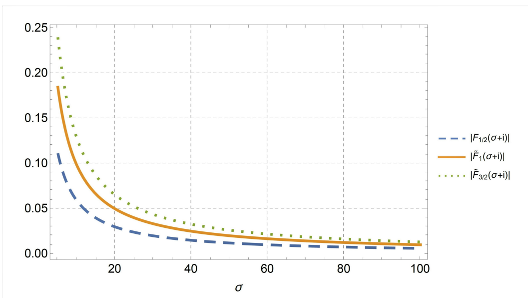

Figure 5. Comparing the fractional LTs $\mathcal{F}_{1/2}$, $\mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $\exp(-t^2)$ - the case of the modulus, assuming $s = \sigma + i$

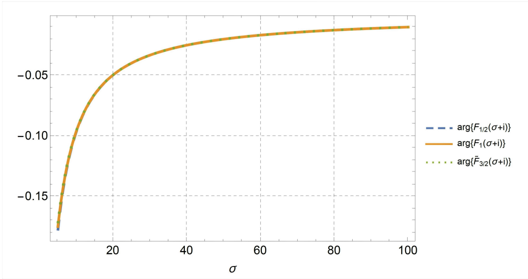

Figure 6. Comparing the fractional LTs $\mathcal{F}_{1/2}, \mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $\exp(-t^2)$ - the case of the argument, assuming $s = \sigma + i$

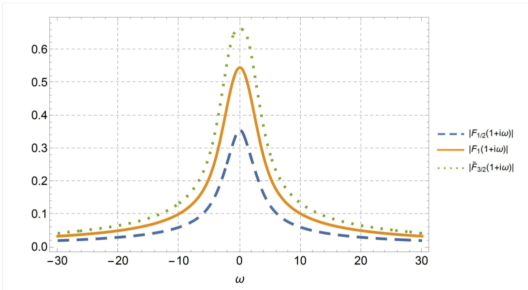

Figure 7. Comparing the fractional LTs $\mathcal{F}_{1/2}, \mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $\exp(-t^2)$ - the case of the modulus, assuming $s = 1 + i\,\omega$

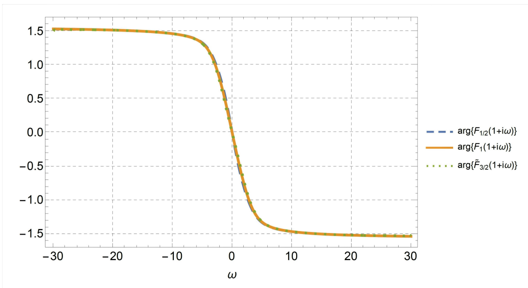

Figure 8. Comparing the fractional LTs $\mathcal{F}_{1/2}, \mathcal{F}_1, \mathcal{F}_{3/2}$, of the function $\exp(-t^2)$ - the case of the argument, assuming $s = 1 + i\,\omega$

5. The reciprocal of the fractional Laguerre-type hyperbolic function

We consider the function $\cosh_{\alpha,1}(a;t)$, where the symbol $a \equiv \{a_n\}$ denote the sequence of coefficients, according to the position

$$\cosh_{\alpha,1}(a;t) =$$
$$1 + a_2 \frac{t^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + a_4 \frac{t^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + \cdots + a_{2n} \frac{t^{2n\alpha}}{[\Gamma(2n\alpha+1)]^2} + \dots . \quad (17)$$

The equation

$$\frac{1}{\cosh_{\alpha,1}(a;t)} =$$
$$b_0 + b_2 \frac{t^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + b_4 \frac{t^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + \cdots + b_{2n} \frac{t^{2n\alpha}}{[\Gamma(2n\alpha+1)]^2} + \ldots \tag{18}$$

in terms of the unknown sequence $b \equiv \{b_n\}$ can be solved by using Bell's polynomials.

Recalling the general result, we find in this case that the solution of the equation

$$\frac{1}{\sum_{n=0}^{\infty} \frac{a^{2n} t^{2n\alpha}}{[\Gamma(2n\alpha+1)]^2}} = \sum_{n=0}^{\infty} \frac{b^{2n} t^{2n\alpha}}{[\Gamma(2n\alpha+1)]^2}\,, \quad \text{i.e.} \quad \cosh_{\alpha,1}(a;t)\, \cosh_{\alpha,1}(b;t) = 1\,,$$

is given, $\forall\, n \geq 0$, by

$$\begin{aligned} b_{2n} = {} & \frac{[\Gamma(2n\alpha+1)]^2}{n!} \sum_{k=0}^{n} (-1)^k\, k!\, a_0^{-(k+1)} \cdot \\ & \cdot B_{n,k}\left(\frac{1!a_2}{[\Gamma(2\alpha+1)]^2}, \frac{2!a_4}{[\Gamma(4\alpha+1)]^2}, \ldots, \frac{(n-k+1)!a_{2(n-k+1)}}{[\Gamma(2(n-k+1)\alpha+1)]^2}\right), \end{aligned} \tag{19}$$

where $B_{2n,k}$ are partial Bell polynomials [22, 23].

In this case we have $a_0 = 1$, and we have to consider the reciprocal of equation (17), i.e.

$$\frac{1}{1 + a_2 \frac{t^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + a_4 \frac{t^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + a_6 \frac{t^{6\alpha}}{[\Gamma(6\alpha+1)]^2} + \ldots} \quad (t \geq 0)\,.$$

Then, according to the above general result, we find

$$\begin{aligned} & \frac{1}{\sum_{n=0}^{\infty} \frac{a_{2n} t^{2n\alpha}}{[\Gamma(2n\alpha+1)]^2}} = \sum_{n=0}^{\infty} \frac{b_{2n} t^{2n\alpha}}{[\Gamma(2n\alpha+1)]^2} = \ = \sum_{n=0}^{\infty} \frac{t^{2n\alpha}}{n!} \sum_{k=0}^{n} (-1)^k\, k! \\ & \times B_{n,k}\left(\frac{1!a_2}{[\Gamma(2\alpha+1)]^2}, \frac{2!a_4}{[\Gamma(4\alpha+1)]^2}, \ldots, \frac{(n-k+1)!a_{2(n-k+1)}}{[\Gamma(2(n-k+1)\alpha+1)]^2}\right). \end{aligned} \tag{20}$$

Therefore, for $\alpha = 1/2$ and $a_n = 1/(n+1)$, the first few values of the b_n coefficients are found to be

$$
\begin{aligned}
b_0 &= 1 \ , \\
b_2 &= -\frac{1}{3} \ , \\
b_4 &= \frac{11}{45} \ , \\
b_6 &= -\frac{29}{105} \ , \\
b_8 &= \frac{191}{525} \ , \\
b_{10} &= -\frac{49}{297} \ , \\
b_{12} &= -\frac{1168697}{315315} \ .
\end{aligned}
$$

5.1. *Hyperbolic Laguerre-type fractional-order Laplace transforms*

Using the above definition of the reciprocal of the fractional Laguerre-type exponential function, we can introduce a fractional-order Laguerre-type Laplace transform by setting

$$
\begin{aligned}
{}_{L_1}\mathcal{HL}_\alpha(f) &:= \int_0^\infty f(t)\,[\cosh_{\alpha,1}(a;st)]^{-1}\,dt = {}_{L_1}\mathcal{HF}_\alpha(s) \\
&= \int_0^\infty f(t)\left[\sum_{n=0}^\infty \frac{(st)^{2n\alpha}}{n!}\sum_{k=0}^n (-1)^k\,k! \right. \qquad (21) \\
&\left. \times B_{n,k}\left(\frac{1!a_2}{[\Gamma(2\alpha+1)]^2}, \frac{2!a_4}{[\Gamma(4\alpha+1)]^2}, \ldots, \frac{(n-k+1)!a_{2(n-k+1)}}{[\Gamma(2(n-k+1)\alpha+1)]^2}\right)\right]\,dt.
\end{aligned}
$$

In what follows, we make a comparison among the Hyperbolic Laguerre-type Laplace Transform of assigned functions and the fractional order Hyperbolic Laguerre-type Laplace transforms of order $\alpha = 1/2$ and $\alpha = 3/2$.

As it is shown in the obtained results, in all cases the graphs of the modulus and argument of the Hyperbolic Laguerre-type Laplace Transform lie between the corresponding graphs of the two considered fractional order Hyperbolic Laguerre-type Laplace transforms. This provides graphical evidence of the monotonicity property satisfied by the fractional order Hyperbolic Laguerre-type Laplace transforms.

6. Numerical Examples

6.1. *Example 1*

Consider the fractional hyperbolic Laguerre-type functions $K(\alpha, \theta)$, defined as

$$K(\alpha, \theta) = \sum_{n=0}^{\infty} \frac{a_n \theta^{2n\alpha}}{[\Gamma(2n\alpha + 1)]^2} \,. \tag{22}$$

In what follows, assuming $a_n = 1/(2n+1)$, the hyperbolic-type fractional Laplace transforms of the function

$$f(t) = \mathrm{ArcSinh}(t/\pi)\,,$$

using as kernels the $[K(\alpha, st]^{-1}$ functions, for $\alpha = 1/4$ and $\alpha = 1/2$, are depicted showing the convergence behavior to the hyperbolic Laguerre-type LT of the considered function for $\alpha = 1$. This has been checked even for other values of α such that $0 < \alpha < 1$, while the same is not true for $\alpha > 1$.

The graphs of the $[K(\alpha, \theta)]^{-1}$ functions for $\alpha = 1/4, 1/2, 1$ are presented in Figure 9.

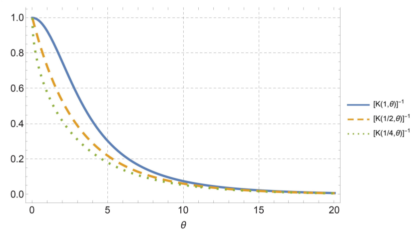

Figure 9. Graphs of the functions $[K(\alpha, \theta)]^{-1}$, for $\alpha = 1/4, 1/2, 1$

Graphs of the modulus and argument of the hyperbolic-type fractional Laplace transforms of the ArcSinh(t/π) functions, for $\alpha = 1/4$ and $\alpha = 1/2$ are depicted in Figures 10-13.

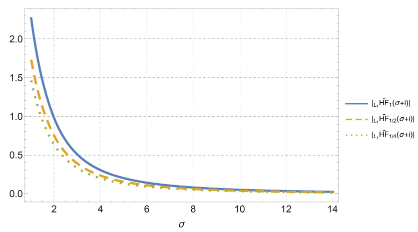

Figure 10. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_1$, of the function ArcSinh(t/π) - the case of the modulus, assuming $s = \sigma + i$

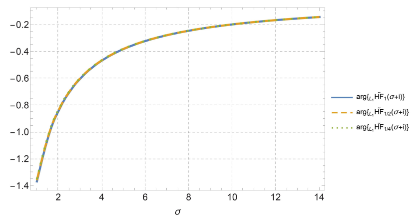

Figure 11. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_{1}$, of the function ArcSinh(t/π) - the case of the argument, assuming $s = \sigma + i$

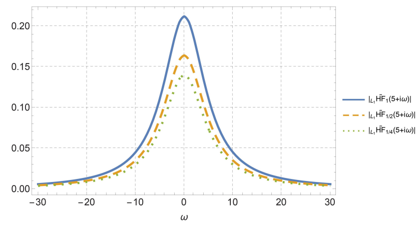

Figure 12. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_{1}$, of the function ArcSinh(t/π) - the case of the modulus, assuming $s = 1 + i\,\omega$

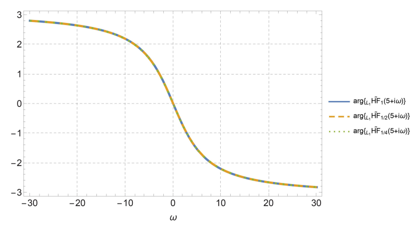

Figure 13. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_{1}$, of the function ArcSinh(t/π) - the case of the argument, assuming $s = 1 + i\,\omega$

6.2. *Example 2*

Consider the fractional hyperbolic Laguerre-type functions $K(\alpha, \theta)$, defined in equation (22), assuming $a_n = \exp(-2n)$.

In what follows the hyperbolic-type fractional Laplace transforms of the function

$$f(t) = \exp(-t^2)\,,$$

using as kernels the $[K(\alpha, st]^{-1}$ functions, for $\alpha = 1/4$ and $\alpha = 1/2$, are depicted showing the convergence behavior to the hyperbolic Laguerre-type LT of the considered function for $\alpha = 1$. This has been checked even for other values of α such that $0 < \alpha < 1$, while the same is not true for $\alpha > 1$.

The graphs of the $[K(\alpha, \theta)]^{-1}$ functions for $\alpha = 1/4, 1/2, 1$ are presented in Figure 14.

Graphs of the modulus and argument of the hyperbolic-type fractional Laplace transforms of the $\exp(-t^2)$ functions, for $\alpha = 1/4$ and $\alpha = 1/2$ are depicted in Figures 15-18.

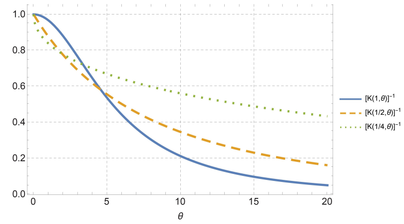

Figure 14. Graphs of the functions $[K(\alpha,\theta)]^{-1}$, for $\alpha = 1/4, 1/2, 1$

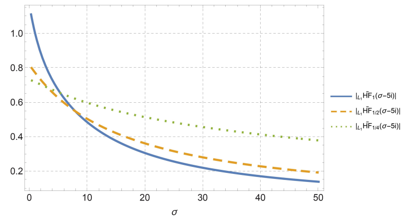

Figure 15. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_{1}$, of the function $\exp(-t^2)$ - the case of the modulus, assuming $s = \sigma + i$

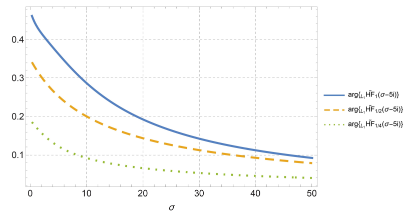

Figure 16. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_{1}$, of the function $\exp(-t^2)$ - the case of the argument, assuming $s = \sigma + i$

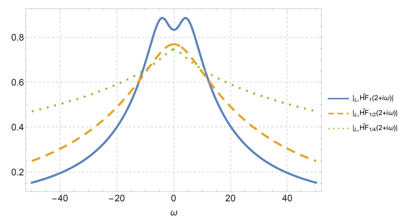

Figure 17. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_{1}$, of the function $\exp(-t^2)$ - the case of the modulus, assuming $s = 1 + i\,\omega$

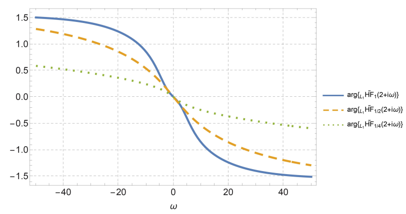

Figure 18. Comparing the hyperbolic-type fractional LTs ${}_{L_1}\mathcal{HF}_{1/4}$, ${}_{L_1}\mathcal{HF}_{1/2}$, ${}_{L_1}\mathcal{HF}_{1}$, of the function $\exp(-t^2)$ - the case of the argument, assuming $s = 1 + i\,\omega$

7. Conclusion

We have shown that, using the Laguerre-type exponentials and their fractional versions, it is possible to define Laguerre-type generalized forms of the classical Laplace transform. We have used a general result to construct the reciprocals of some exponential-type functions, and we have used these reciprocals in place of the kernel of the usual Laplace transform.

Several worked examples of the new transformations, computed using the computer algebra system Mathematica© have been reported in the preceding Sections.

The introduced transformations could be used in the framework of fractional differential equations or in that of the Laguerre-type ones.

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