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Two Methods for the Numerical Solution of a System of Nonlinear Partial Differential Equations

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This study investigates a comparative analysis of approximate solutions for the initial-boundary value problem of a two-dimensional nonlinear model. The analysis employs two distinct numerical approaches: the variable direction difference scheme and the averaged sum approximation method.

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Nonlinear partial differential equations describe numerous real-world processes, including a important mathematical model for vein formation in plant leaves, proposed by Mitchison as follows [1]:

$$\begin{aligned} \frac{\partial s}{\partial t} &= \frac{\partial}{\partial x}\left(D_1\frac{\partial s}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_2\frac{\partial s}{\partial y}\right), \\ \frac{\partial D_1}{\partial t} &= f_1\left(D_1, D_1\frac{\partial s}{\partial x}\right), \\ \frac{\partial D_2}{\partial t} &= f_2\left(D_2, D_2\frac{\partial s}{\partial y}\right), \end{aligned} \tag{1}$$

where $s(x,y)$ is the concentration, $D_1(x,y)$ and $D_2(x,y)$ are the diffusion coefficients for flux parallel to the x and y axes respectively, f_1, f_2 are given functions of their arguments.

Investigations for one-dimensional analog of system (1) are carried out in [2]. Many mathematical models describing the similar diffusion processes are also presented and discussed in various papers (see, for example, [2]-[7] and references

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therein). In the direction of biological modeling it is necessary to note the following work [8].

Beginning from the basic works [9]-[11] the methods of constructing of effective algorithms for the numerical solution of the multi-dimensional problems of the mathematical physics and the class of problems solvable with the help of these algorithms were essentially extended. Currently there are some effective algorithms for solving the multi-dimensional problems (see, for example, [12]-[14] and references therein). These algorithms mainly belong to the methods of splitting-up or sum approximation. Some schemes of the variable directions are constructed and studied in the following works [15]-[20].

In the presented article two different approaches were applied to construct approximate solutions of the initial-boundary value problem for the system of equations corresponding to a two-dimensional nonlinear model (1). The first approach, described in the presented note, uses a decomposition method based on an averaged model studied in [3]. An appropriate scheme and necessary algorithms for computer implementation were built as well as the program codes were compiled. Numerical experiments were conducted for various tests and some results of those experiments are presented here. The second approach uses a variable direction difference scheme [4]. Necessary algorithms for computer realization were built for this method too. The number of operations was determined for both methods. The time required for the execution of the algorithms and the accuracy of the numerical experiments have been compared for both approaches. An analysis of the obtained results was carried out, and appropriate conclusions were done.

In $Q = \overline{\Omega} \times T = [0,1] \times [0,1] \times [0,T]$, where T is positive constant, let us consider the following initial-boundary value problem for the two-dimensional system based on general model (1):

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x_1}\left(V_1 \frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(V_2 \frac{\partial U}{\partial y}\right), \\ \frac{\partial V_1}{\partial t} &= -V_1 + g_1\left(V_1 \frac{\partial U}{\partial x}\right), \\ \frac{\partial V_2}{\partial t} &= -V_2 + g_2\left(V_2 \frac{\partial U}{\partial y}\right), \end{aligned} \tag{2}$$

with initial

$$\begin{aligned} U(x,y,0) &= U_0(x,y), \\ V_1(x,y,0) &= V_{10}(x,y), \\ V_2(x,y,0) &= V_{20}(x,y), \end{aligned} \tag{3}$$

and boundary conditions

$$\begin{aligned} U(x,0,t) &= U(x,1,t), \\ U(0,y,t) &= U(1,y,t), \\ t \in [0,T], &\quad (x,y) \in [0,1]. \end{aligned} \tag{4}$$

Here $g_\alpha, U_0, V_{\alpha 0}, \alpha = 1,2$ are given sufficiently smooth functions satisfying the following conditions:

$$V_{\alpha 0} \geq \sigma_0 \quad \sigma_0 = const > 0, \quad (x,y) \in \overline{\Omega},$$
$$g_0 \leq g_\alpha \leq G_0, \quad |g_\alpha^{'}| \leq G_1, \quad \xi_\alpha \in R, \quad \alpha = 1,2, \tag{5}$$

where g_0, G_0, G_1 in (5) are some positive constants.

This study explores the numerical solution of the problem (2)-(4) using the following two approaches: a variable direction difference scheme and a difference scheme corresponding to the average method.

We will use the following known notations for the grid construction on domain Q:

$$\overline{\omega}_h = \{(x,y) = (ih, jh), \quad i,j = 0,1,...,M, \quad Mh = 1\},$$
$$\overline{\omega}_{1h} = \{(x,y) = ((i-1/2)h, jh) \quad i = 1,2,...,M, \quad j = 0,1,...,M, \quad Mh = 1\},$$
$$\overline{\omega}_{2h} = \{(x,y) = (ih, (j-1/2)h) \quad i = 0,1,...,M, \quad j = 1,2,...,M, \quad Mh = 1\},$$
$$\omega_h = \Omega \cap \overline{\omega}_h, \quad \gamma_h = \overline{\omega}_h / \omega_h, \quad \overline{\omega}_h = \omega_h \cup \gamma_h,$$
$$\omega_\tau = \{t_k = k\tau, \quad k = 0,...,K, \quad K\tau = T\},$$
$$\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau, \quad \overline{\omega}_{\alpha h\tau} = \overline{\omega}_{\alpha h} \times \omega_\tau, \quad \alpha = 1,2.$$

Following to the known notations [13], in equations below the difference scheme of variable directions is presented for the problem (2)-(4):

$$u_{1t} = \left(\widehat{v}_1 \widehat{u}_{1\overline{x}}\right)_x + \left(\widehat{v}_2 \widehat{u}_{2\overline{y}}\right)_y,$$
$$u_{2t} = \left(\widehat{v}_1 \widehat{u}_{1\overline{x}}\right)_x + \left(\widehat{v}_2 \widehat{u}_{2\overline{y}}\right)_y,$$
$$v_{1t} = -\widehat{v}_1 + g_1\left(v_1, u_{1\overline{x}}\right),$$
$$v_{2t} = -\widehat{v}_2 + g_2\left(v_2, u_{2\overline{y}}\right),$$
$$u_1(x,y,0) = u_2(x,y,0) = U_0(x,y), \quad (x,y) \in \overline{\omega}_h, \tag{6}$$
$$v_1(x,y,0) = V_{10}(x,y), \quad (x,y) \in \overline{\omega}_{1h},$$
$$v_2(x,y,0) = V_{20}(x,y), \quad (x,y) \in \overline{\omega}_{2h},$$
$$u_1(x,t,t) = u_2(x,y,t) = 0, \quad (x,y,t) \in \gamma_h \times \omega_\tau.$$

To find an approximate solution on a new layer in the above-mentioned scheme, system of equations is solved in sequence for $\widehat{v}_1, \widehat{u}_1$ functions and then for functions $\widehat{v}_2, \widehat{u}_2$ by using solutions for $\widehat{v}_1, \widehat{u}_1$. The final approximate solutions for the target layer are $\widehat{u}_2, \widehat{v}_1, \widehat{v}_2$. After moving to a new layer, the initial conditions for v_1, v_2, u_1 and u_2 functions are defined by the solutions that were derived from the previous layer.

For describing the second approach of numerical solution of the problem (2)-(4), let us consider the following difference scheme corresponding to the average method:

$$
\begin{gathered}
u_{1t} = \left(\widehat{v}_1 \widehat{u}_{1\overline{x}}\right)_x, \quad u_{2t} = \left(\widehat{v}_2 \widehat{u}_{2\overline{y}}\right)_y, \\
v_{1t} = -\widehat{v}_1 + g_1\left(v_1, u_{1\overline{x}}\right), \quad v_{2t} = -\widehat{v}_2 + g_2\left(v_2, u_{2\overline{y}}\right), \\
u_1(x,y,0) = u_2(x,y,0) = U_0(x,y), \quad (x,y) \in \overline{\omega}_h, \\
v_1(x,y,0) = V_{10}(x,y), \quad (x,y) \in \overline{\omega}_{1h}, \\
v_2(x,y,0) = V_{20}(x,y), \quad (x,y) \in \overline{\omega}_{2h}, \\
u_1(x,t,t) = u_2(x,y,t) = 0, \quad (x,y,t) \in \gamma_h \times \omega_\tau.
\end{gathered} \tag{7}
$$

To find approximate solutions at all points in $t = t_k$ layer, (u_1, v_1) and (u_2, v_2) approximate solutions at these points are found at the same time from the above-mentioned scheme. The final approximate solution for the current layer is u, v_1, v_2 where $u = \eta_1 u_1 + \eta_2 u_2$, $\eta_1 \eta_2 > 0$, $\eta_1 + \eta_2 = 1$. After moving to a new layer, the initial conditions for v_1 and v_2 functions are defined by the solutions from the previous layer, while the initial conditions for u_1 and u_2 functions are defined by function u that was derived from the same previous layer.

Table 1. CPU time and error for solution U, V_1, V_2 for variable directions difference scheme (6).

t	CPU Time	ERRORS		
		U	V_1	V_2
0.2	2.74E-01	1.08E-04	2.87E-05	1.66E-07
0.4	4.83E-01	3.90E-04	2.18E-05	1.22E-06
0.6	6.83E-01	3.70E-04	1.80E-05	5.16E-06
0.8	8.72E-01	8.67E-05	3.72E-05	9.11E-06
1.0	1.08E+00	7.14E-15	7.98E-05	9.53E-06

Examination of the algorithms presented in numerical methods can be carried out using different parameters but there are usually two important factors: accuracy and computation time.

In our case, to compare the accuracy of the above-mentioned methods, the maximum absolute values of the differences at the same points are selected and compared to each other.

In the following experiment, the right side is chosen so that the exact solution is:

$$
\begin{aligned}
U(x,y,t) &= xy(1-x)(1-y)(1+t), \\
V_1(x,y,t) &= xy(1-x)(1-y)(2+t+t^2)+1, \\
V_2(x,y,t) &= xy(1-x)(1-y)(2+t+t^3)+1.
\end{aligned}
$$

We have chosen function $g_\alpha(\xi)$ as follows

$$
g_\alpha(\xi) = \tfrac{1}{1+(1+\xi)^2}, \quad \alpha = 1, 2.
$$

Table 2. CPU time and error for solution U, V_1, V_2 for average method (7).

t	CPU Time	ERRORS		
		U	V_1	V_2
0.2	2.24E-01	9.14E-04	2.39E-04	2.33E-05
0.4	4.02E-01	7.82E-04	5.43E-04	5.30E-05
0.6	5.84E-01	5.66E-04	9.21E-04	2.98E-05
0.8	7.66E-01	6.03E-04	1.38E-03	3.44E-05
1.0	9.38E-01	1.33E-14	1.94E-03	8.77E-05

Numerical experiments allow us to conclude that the averaged method can be executed faster than the variable directions difference scheme. This fact was expected because of the averaged model algorithm computes data on a new layer in parallel, while in variable directions difference scheme it goes in sequence. Also, the number of operations is less in the case of the averaged model scheme compared to the variable directions difference scheme.

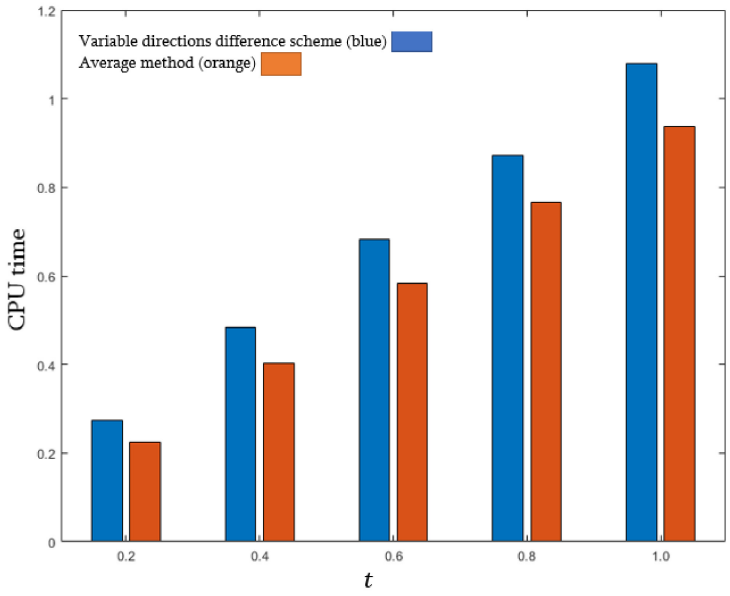

Figure 1. CPU time for variable directions difference scheme and average method. Variable directions difference scheme (blue), Average method (orange)

By analyzing the maximum discrepancies between exact and approximate solutions on the same layer for both methods, the variable direction difference scheme demonstrates better accuracy compared to the corresponding averaged method difference scheme. This can be explained by the inherent nature of difference schemes. In the variable directions approach, the equations governing the U function retain their two-dimensionality. Conversely, the averaged method reduces the U function to a system of one-dimensional equations, potentially introducing a loss of accuracy.

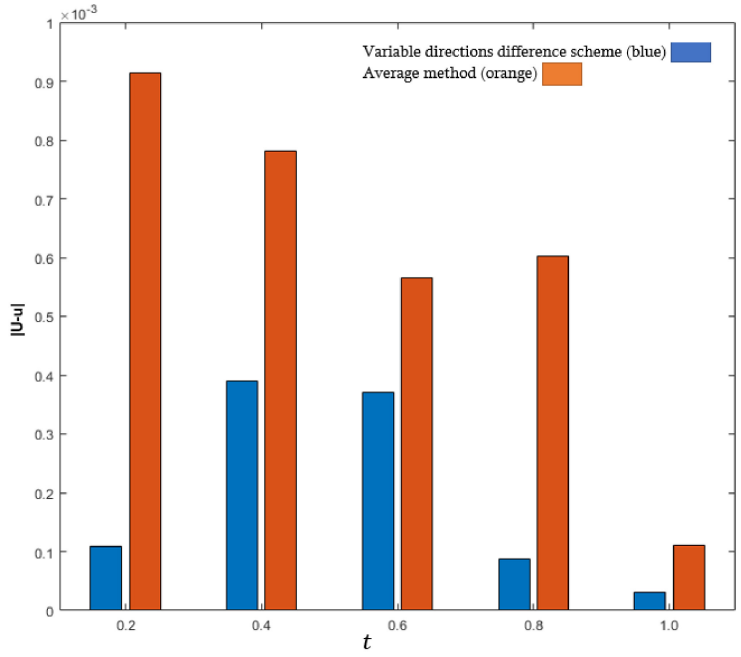

Figure 2. Errors for U using variable directions difference scheme and average method.

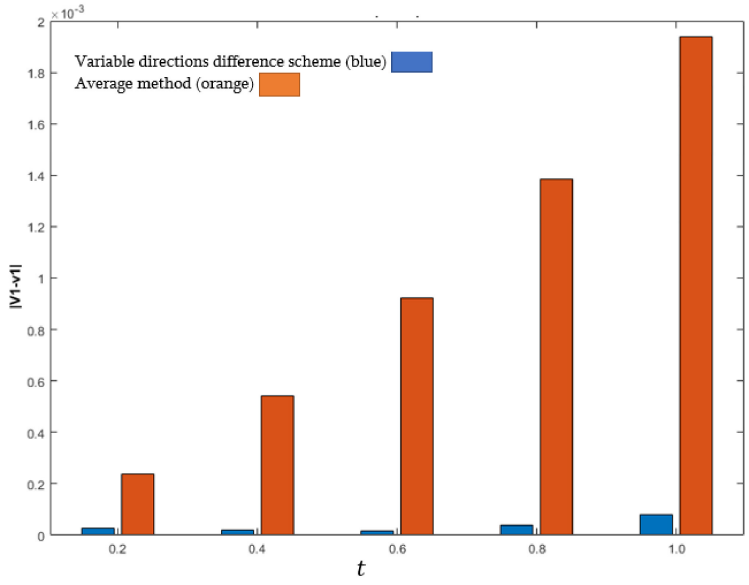

Figure 3. Errors for V_1 using variable directions difference scheme and average method.

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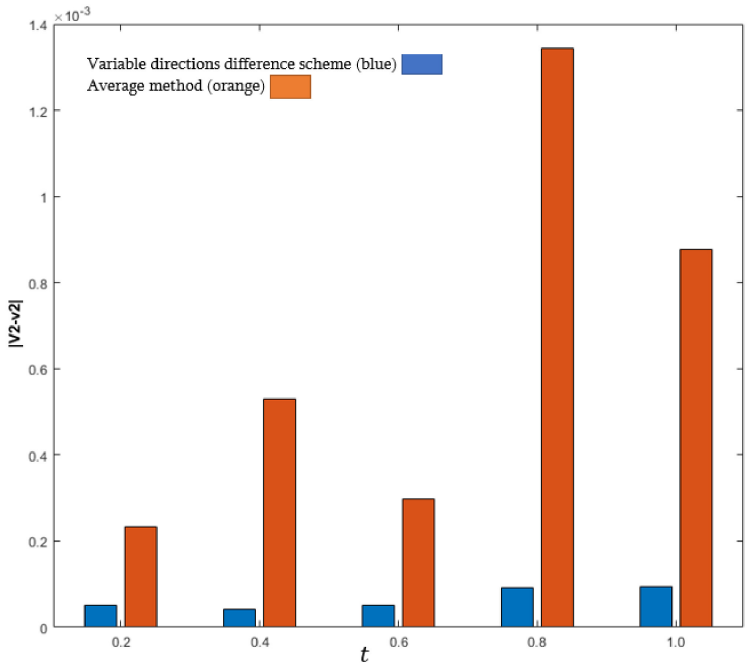

Figure 4. Errors for V_2 using variable directions difference scheme and average method.

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