Bulletin of TICMI
Vol. 28, No. 2, 2024, 139–142

On Absolutely Negligible Uniform Sets

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(*Received December 5, 2023; Revised October 1, 2024; Accepted October 16, 2023*)

We consider absolutely negligible uniform subsets of the Euclidean plane R^2. Furthermore, it is shown that there exist 2^c many R^2-absolutely negligible uniform subsets in the Euclidean plane R^2.

Keywords: Uniform subset, negligible set, absolutely nonmeasurable set, Hamel basis.

AMS Subject Classification: 28A05, 28D05, 03E20.

Intensive studies in mathematical analysis, in the theory of functions of a real variable, and in Lebesgue measure theory inspired and stimulated further research of various paradoxical point sets on the real line R, and more generally, in the Euclidean space R^n where $n \geq 1$.

In this paper we consider well-known point sets in the Euclidean plane R^2 - so called uniform sets and study the family of such sets and their properties. The study of uniform sets was started with one intersting problem of Luzin, posed by him many years ago (see [5]). This problem has a close connection with the famous Sierpinski partition of the plane R^2 (see [6], [7]).

First of all, let us recall the formulation of this beautiful problem. Namely, Luzin asked whether there exists a function

$$f : R \to R$$

such that the whole plane R^2 can be covered by countably many isometric copies of the graph of f.

It should be mentioned here that under the Continuum Hypothesis CH, this problem admits a positive solution even in a much stronger form. Actually, a positive answer to the Luzin question follows from the existence of a Sierpinski type partition for the Euclidean plane R^2 (see [4]).

The solution of this problem, within the theory ZFC was obtained some time

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ISSN: 1512-0082 print

later by Davies (see his work [9]).

Definition 1: Let X be a subset of the plane R^2 and let p be a straight line in R^2. This X is called a uniform subset of R^2 with respect to p if for each line $p' \subset R^2$ parallel to p, we have

$$card(p' \cap X) = 1.$$

Obviously, for a set $X \subset R^2$ and for any two parallel straight lines p and q in R^2, the following assertions are equivalent:

- X is uniform with respect to p;
- X is uniform with respect to q.

Further, we shall say that a set $X \subset R^2$ is uniform if there exists at least one straight line p in R^2 such that X is uniform with respect to p.

It is not hard to observe that the next two assertions are also equivalent:

(1) there exists a function $f : R \to R$ such that the plane R^2 can be covered by countably many isometric copies of the graph of f;
(2) the plane R^2 can be covered by countably many uniform subsets of R^2.

Therefore, the Luzin problem is reduced to the problem of the existence of a countable family of uniform sets, whose union is identical with R^2.

Let p be an arbitrary straight line in R^2 and let z be a point of R^2. We denote by $p(z)$ the unique line in R^2 which is parallel to p and contains z.

A countable family $P = (p_k)_{k<\omega}$ of straight lines in R^2 will be called admissible if all these lines are pairwise distinct and all of them contain the origin of R^2.

In fact, Davies was able to prove the following key lemma (see [9]).

Lemma 1: *Let $P = (p_k)_{k<\omega}$ be an admissible family of straight lines in R^2. Then there exists a family $(Z_k)_{k<\omega}$ of subsets of R^2 satisfying the following two relations:*

(1) each set Z_k is uniform with respect to the line p_k;
(2) $\cup\{Z_k : k < \omega\} = R^2$.

Let e be a nonzero vector in the plane R^2.

Definition 2:

(1) A set $A \subset R^2$ is called *finite* in direction e if $card(l \cap A) < \omega$ for any straight line $l \subset R^2$ parallel to e;
(2) A set $B \subset R^2$ is called *countable* in direction e if $card(l \cap B) \leq \omega$ for any straight line $l \subset R^2$ parallel to e.

Example 1: A Mazurkiewicz subset of the Eucliden plane R^2 (which was constructed by Mazurkiewicz in 1914) is any point set M in R^2, such that every straight line in the plane meets M in exactly two points. So it is natural to say that a set $Z \subset R^2$ is a Mazurkiewicz subset of R^2 if $card(Z \cap l) = 2$ for every straight line l lying in R^2 (cf. [2]).

The above definition immediately implies that, for any nonzero vector $e \in R^2$, the Mazurkiewicz set Z is finite in direction e.

Lemma 2: *Let e be a nonzero vector in R^2 and let Z be a subset of R^2 countable in direction e. Then there exist a set $Z_0 \subset R^2$ and a countable family $\{h_n : n <$*

$\omega\} \subset R^2$ such that:

(1) Z_0 is uniform in direction e;
(2) $Z \subset \cup\{h_n + Z_0 : n < \omega\}$.

Let $M(R^2)$ denote the class of all nonzero σ-finite translation invariant measures on the plane R^2.

A set $X \subset R^2$ is called *negligible* with respect to $M(R^2)$ (or, briefly, R^2-negligible) if these two conditions are satisfied for X:

- there exists a measure $\nu \in M(R^2)$ such that $X \in dom(\nu)$;
- for any measure $\mu \in M(R^2)$, the relation $X \in dom(\mu)$ implies the equality $\mu(X) = 0$.

Undoubtedly, negligible sets are of interest for the general theory of invariant (quasi-invariant) measures. However, the notion of an absolutely negligible set presented below is more delicate and useful than the notion of negligible set (cf. [1], [4]).

A set $Y \subset R^2$ is called *absolutely negligible* with respect to $M(R^2)$ (or, briefly, R^2-absolutely negligible) if, for every measure $\mu \in M(R^2)$, there exists a measure $\mu' \in M(R^2)$ extending μ and such that the relations $Y \in dom(\mu')$ and $\mu'(Y) = 0$ hold true.

Let us remark, that any R^2-absolutely negligible set is also R^2-negligible, but the converse assertion fails to valid.

Lemma 3. *If a set $Z \subset R^2$ is finite in some direction l, then Z is negligible with respect to the class $M(R^2)$.*

Example 2:

- every Hamel basis of the space R^n is an R^n-absolutely negligible subset of R^n (see [3]);
- since any Mazurkiewicz subset Z of the plane is finite in every direction l, one can conclude that Z is negligible with respect to the class $M(R^2)$.

As mentioned in Example 2, every Mazurkiewicz set turns out to be R^2-negligible. Moreover, there exists a measure μ on R^2 which extends the Lebesgue measure λ_2, is invariant under the group of all isometries of R^2, and contains in its domain all Mazurkiewicz subsets of R^2 (see [8]).

The following lemma is crucial for obtaining the main result of this paper.

Lemma 4: *There exists a Mazurkiewicz subset of R^2 which is a Hamel basis of R^2.*

A detailed proof of this lemma can be found in [8].

From Lemma 4 the next statement readily follows.

Lemma 5: *There exists a Mazurkiewicz subset X of R^2 which is absolutely negligible with respect to the class $M(R^2)$.*

Using Lemmas 4 and 5, we obtain the following statement.

Theorem 1: *Let c denote the cardinality of the continuum. There exist 2^c many R^2-absolutely negligible uniform subsets in the Euclidean plane R^2.*

The behavior of absolutely negligible sets under some natural algebraic operations (e.g., under algebraic sums) was studied in several works (see, for instance, [1] and [10]).

Acknowledgments

The work was supported by the Shota Rustaveli National Science Foundation of Georgia (SRNSFG), Grant YS-21-1667.

References

[1] M. Beriashvili, M. Khachidze, A. Kirtadze. *Absolutely negligible sets and their algebraic sums*, **177**, 1 (2023), 131-133

[2] J.J. Dijkstra, J. van Mill. *Two point set extension - a counterexample*, Proc. Amer. Math. Soc., **125**, 8 (1997)

[3] A. Kharazishvili. *One property of Hamel bases*, Bull. Acad. Sci. GSSR, **95**, 2 (1979)

[4] A. Kharazishvili. *Questions in the Theory of Sets and in Measure Theory*, TSU, Tbilisi, 1978

[5] N.N. Luzin. *Collected Works,* Izd. Akad. Nauk SSSR, Moscow, 1958

[6] W. Sierpinski. *Cardinal and Ordinal Numbers*, PWN, Warszawa, 1958

[7] W. Sierpinski. *Hypothes de Continu*, Monografie Matematyezne, **4** Warszawa-Lwow, 1934

[8] A. Kharazishvili. *On Mazurkiewicz Sets from the Measure-Theoretical Point of View*, Bulletin of TICMI, **21**, 1 (2017), 45-54

[9] R.O. Davies. *Covering the plane with denumerably many curves*, J. Lond. Math. Soc., **38** (1963), 433-438

[10] A. Kharazishvili, A. Kirtadze. *On algebraic sums of measure zero sets in uncountable commutative groups*, Proceedings of A. Razmadze Mathematical Institute, **135** (2004), 97-103