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The Consistent Criterion for Hypotheses Testing

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In this paper, for a statistical structure associated with a counting measure, the sufficient and necessary conditions for the admitting the constent criteria for hypothesis testing are given.

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1. Introduction

Based on the selection, it is possible to formulate mutually exclusive hypotheses about the theoretical distribution, one of which should be preferred to the others. One of several hypotheses selection task is solved by a statistical criterion formation. Conclusions about the distribution, as a rule, contain certain errors. From the needs of the mathematical statistics theory, the question often arises concerning the probability of transition from an orthogonal or from a weakly separable statistical structure to a strongly separable statistical structure.

Z. Zerakidze defined and studied the consstent criteria for hypotheses testing (see [2], [3], [5]), which will be referred to below as the Z. Zerakidze criterion. Using these criteria we accept infallible conclusions for an infinite number of hypotheses with probability of one using Z. Zerakidze's criteria for hypotheses testing, the probability of an error of any kind is equal to zero.

2. Z. Zerakidze's criterion for hypotheses testing

Let (E, S) be a measurable space with a given family of probability measures: $\{\mu_h, h \in H\}$.

The following definitions are taken from [1] - [5].

Definition 2.1: An object $\{E, S, \mu_h, h \in H\}$ is called a statistical structure.

Definition 2.2: A statistical structure $\{E, S, \mu_h, h \in H\}$ is called orthogonal (singular) if a family of probability measures $\{\mu_h, h \in H\}$ constists of pairwise singular measures (i.e. $\mu_{h'} \perp \mu_{h''}, \forall h^{'} \neq h^{''}$).

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Definition 2.3: A statistical structure $\{E, S, \mu_h, h \in H\}$ is called weakly separable (WS) if there exists a family of S-measurable sets $\{X_h, h \in H\}$ such that

$$\mu_h(X_{h'}) = \begin{cases} 1, & if\ h = h'; \\ 0, & if\ h \neq h' \end{cases} \qquad (h, h' \in H).$$

Definition 2.4: A statistical structure $\{E, S, \mu_h, h \in H\}$ is called separable (S) if there exists a family of S-measurable sets $\{X_h, h \in H\}$ such that

1) $\mu_h(X_{h'}) = \begin{cases} 1, & if\ h = h'; \\ 0, & if\ h \neq h' \end{cases} \qquad (h, h' \in H);$

2) $\forall h, h' \in H: \ \ card(X_h \cap X_{h'}) < c, \ if\ h \neq h',$

where c denotes the cintinuum power.

Definition 2.5: A statistical structure $\{E, S, \mu_h, h \in H\}$ is called strongly separable (SS) if there exists a disjoint family of S-measurable sets $\{X_h, h \in H\}$ such that the following relations are fulfilled:

$$\mu_h(X_h) = 1, \ \ \forall h \in H.$$

Example 2.6 Let $E = R \times R$ (where $R = (-\infty, +\infty)$) and let $S = B(R \times R)$ be a Borel σ-algebra of subsets of $R \times R$. Let's take the S-measurable sets

$$X_h = \{-\infty < x < +\infty, \ y = h, \ if \ \ h \in (0, +\infty)\}$$

and assume that

$$\mu_h(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2h^2}} dx$$

are linear Gaussian measures on X_h, $h \in (0, +\infty)$. Then the statistical structure $\{R \times R, \ S, \ \mu_h, \ h \in (0, +\infty)\}$ is strongly separable.

Example 2.7 Let $E = R \times R$ and $S = B(R \times R)$ be a Borel σ-algebra of subsets of $R \times R$. Let's take the S-measurable sets

$$X_h = \begin{cases} -\infty < x < +\infty, \ y = h, \ if \ \ h \in R; \\ x = h, \ -\infty < y < +\infty, \ if \ \ h \in R. \end{cases}$$

Assume that μ_h are linear Gaussian measures on X_h, $h \in R$. Then the statistical structure $\{R \times R, \ S, \ \mu_h, \ h \in R\}$ is separable, but not strongly separable.

Example 2.8 Let $E = R \times R \times R$, let S be a Borel σ-algebra of subsets on E. Let's take the S-measurable sets

$$X_h = \begin{cases} -\infty < x < +\infty, \ -\infty < y < +\infty, \ z = h, \ if \ \ h \in R; \\ x = h, \ -\infty < y < +\infty, \ -\infty < z < +\infty, \ if \ \ h \in R; \\ -\infty < x < +\infty, \ y = h, \ -\infty < z < +\infty, \ if \ \ h \in R \end{cases}$$

and assume that μ_h are plane Gaussian measures on X_h. Then the statistical structure $\{R \times R \times R, \ S, \ \mu_h, \ h \in R\}$ is weakly separable, but not separable.

Example 2.9 Let $E = R \times R$, S be a Borel σ-algebra of subsets of $R \times R$. Let's take the S-measurable sets

$$X_h = \begin{cases} -\infty < x < +\infty, \ y = h, \ if \ \ h \in R \setminus \{0\}; \\ -\infty < x < +\infty, \ -\infty < y < +\infty, \ if \ \ h = 0 \end{cases}$$

and assume that $\mu_h, \ h \in R \setminus \{0\}$, are linear Gaussian measures on X_h and μ_0 is a plane Gaussian measure on $R \times R$. Then the statistical structure $\{R \times R, \ S, \ \mu_h, \ h \in R\}$ is orthogonal, but not weakly separable.

Definition 2.10: We consider the concept of the hypothesis as any assumption that determines the form of the distribution of population.

Let H be the set of hypotheses and let $B(H)$ be σ-algebra of subsets of H which contains all finite subsets of H.

Definition 2.11: A statistical criterion is any measurable mapping

$$\delta \ : \ (E, S) \longrightarrow (H, B(H)).$$

Definition 2.12: We will say that the statistical structure $\{E, S, \mu_h, h \in H\}$ admits Z. Zerakidze's criterion for hypothesis testing if there exists at least one measurable mapping

$$\delta \ : \ (E, S) \ \longrightarrow \ (H, B(H)),$$

such that

$$\mu_h(\{x : \delta(x) = h\}) = 1, \ \ \forall \, h \in H.$$

Definition 2.13: The probability $\alpha_h(\delta) = \mu_h(\{x : \delta(x) \neq h\})$ is called the probability of error of the h-th type for a given criterion δ.

The decisive role in all the applied measures is played by additivity with respect to certain partitions of the space E and all those subsets of E, that are measurable with respect to these measures. However, for the most measures there are non-measurable sets (see [1]). With respect to each measure of μ on (E, S), the sets $\emptyset$ and E are always μ-measurable. It may happen, that only they are μ-measurable sets. Whether there are such measures on (E, S) that all subsets of the space E are measurable with respect to these measures. For each set of E is denoted by 2^E class of all subsets of the set E.

Definition 2.14: To each set of A from 2^E we put into correspondence the cardinality of the set A, i.e.

$$\mu(A) = cardA, \ \ \forall A \in 2^E.$$

Such μ measure is called a counting measure.

Remark 1 : 1) All subsets of the space E are measurable with respect to counting measure; 2) If A contains m finite elements, then $\mu(A) = cardA = m$; 3) If A is an infinite set, then $\mu(A) = +\infty$.

Remark 2 : 1) If E is a finite set, then the counting measure μ is finite, e.g. $E = \{b_1, b_2, b_3, b_4\}$, then $\mu(E) = card\{b_1, b_2, b_3, b_4\} = 4$; 2) If E is an infinite set, whose cardinality is greater that or equal to the continuum, then the counting measure μ is infinite, but is not σ-finite, e.g. if $E = R$, then the counting measure on R is infinite, but is not σ-finite, because R can not be imagined as $R = A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots$ so that $A_i \cap A_j = \emptyset$, $\forall i \neq j$ and $cardA_i < +\infty$, $\forall i \in N$.

Theorem 2.15 : *If the counting measure μ is defined on $N = \{1, 2, ..., n, ...\}$ the set of natural numbers, then this measure μ is σ-finite.*

Proof : Let the set of N be represented in the following form

$$N = \{1\} \cup \{2\} \cup \cdots \cup \{n\} \cup \cdots.$$

It's clear, that these sets are pairwise non-interesting and counting measure μ of these sets is finite and $\mu(\{1\}) = \mu(\{2\}) = \mu(\{n\}) = \cdots = 1$; whereas, the counting measure μ on N is equal to $\mu(N) = cardN = +\infty$.

Thus, the counting measure on N is σ-finite. □

Theorem 2.16 : *If the counting measure μ is defined on $(N, B(N))$, then for this measure μ there exists a probability measure $\widetilde{\mu}$ on $(N, B(N))$, such that $\mu \sim \widetilde{\mu}$, i.e. $\mu \ll \widetilde{\mu}$ and $\widetilde{\mu} \ll \mu$ (measures μ and $\widetilde{\mu}$ are equivalent)*

Proof : Since the counting measure μ on $(N, B(N))$ is σ-finite, it follows that, the set N can be represented as a countable union of pairwise non-intersecting sets $N = \{1\} \cup \{2\} \cup \cdots \cup \{n\} \cup \cdots$.

Let's choose a sequence of positive numbers $\rho_k = 1/2^k$, so as $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ and put

$$\widetilde{P}(B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \mu(B \cap \{k\}),$$

then

$$\widetilde{\mu}(N) = \sum_{k=1}^{\infty} \frac{1}{2^k} \mu(N \cap \{k\}) = 1,$$

i.e. $\widetilde{\mu}$ is probability measure.

It's easy to prove, that $\widetilde{\mu}(B) = 0$, then $\mu(B) = 0$ and vice versa, i.e. μ is equivalent to $\widetilde{\mu}$. This theorem has been proved. □

Let's consider a measurable space $(N, B(N))$. Hence, it follows that certain counting measure here is σ-finite, i.e. $N = \{1\} \cup \{2\} \cup \cdots \cup \{n\} \cup \cdots$ and on each set $\{n\}$ the counting measure is $\mu(\{n\}) = 1$. Let's define probability measures on $(N, B(N))$ as follows

$$\mu_n = \mu(n), \quad \forall n \in N.$$

Then we get the statistical structure $\{N, B(N), \mu_n, n \in N\}$.

Let $H = N = \{1, 2, ..., n, ...\}$ be a set of hypotheses

Theorem 2.17: *In order that the statistical structure $\{N, B(N), \mu_n, n \in N\}$ being associated with counting measure to admit Z. Zerakidze's criterion for hypotheses testing it is necessary and sufficient that this statistical structure was strongly separable.*

Proof: Necessity. Since the statistical structure $\{N, B(N), \mu_n, n \in N\}$ admits Z. Zerakidze's criterion for hypotheses testing, there exists a measurable mapping $\delta : (N, B(N)) \longrightarrow (N, B(N))$ such that $\mu_n(\{x : \delta(x) = n\}) = 1$, $\forall n \in N$. Let's denote $X_n = \{x : \delta(x) = n\}$ for $n \in N$. Then, we obtain that:

1) $\mu_n(X_n) = \mu_n(\{x : \delta(x) = n\}) = 1, \ \forall n \in N$;
2) $X_{n_1} \cap X_{n_2} = \{x : \delta(x) = n_1\} \cap \{x : \delta(x) = n_2\} = \emptyset, \ \forall n_1 \neq n_2$;
3) $\cup_{n \in N} X_n = \cup_{n \in N} \{x : \delta(x) = n\} = N$.

Therefore the statistical structure$\{N, B(N), \mu_n, n \in N\}$ is strongly separable. The necessity has been proved.

Sufficiency. Since the statistical structure$\{N, B(N), \mu_n, n \in N\}$ is strongly separable, there exists a family of elements of σ-algebra $B(N)$ such that:

1) $\mu_n(X_n) = 1, \ \forall n \in N$;
2) $X_{n_1} \cap X_{n_2} = \emptyset, \ \forall n_1 \neq n_2 \ (n_1, n_2 \in N)$;
3) $\cup_{n \in N} X_n = N$.

For $x \in N$, we put $\delta(x) = n$, where n is the unique hypothesis from the set N for which $x \in X_n$. The existence of such a unique hypothesis from N can be proved using conditions 2), 3).

Since $B(N)$ contains all finite subsets of N, we conclude that

$$\mu_n(\{x : \delta(x) = n\}) = \mu_n(X_n) = 1, \ \forall n \in N.$$

This is easily proven by the next theorem. □

Theorem 2.18: *The statistical structure $\{N, B(N), \mu_n, n \in N\}$ associated with the counting measure admits Z. Zerakidze's criterion for hypotheses testing if and only if the probability of error of all kinds is equal to zero for the criterion δ.*

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