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On the Test Results of a Method of Solution of the Nonlinear Integro-Differential Equation of String Oscillation

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We consider numerical algorithm for Kirchhoff type inhomogeneous integro-differential equation describing the string oscillation. The algorithm has been approved by tests and the results of calculations are presented in tables and graphs.

Keywords: Kirchhoff string wave equation, Galerkin's method, Crank-Nicolson difference scheme, Jacobi iteration process, Test results.

AMS Subject Classification: 35L20, 65H10, 65M60, 65N06, 74G15.

1. Statement of problem

Consider the nonlinear inhomogeneous equation

$$w_{tt}(x,t) - \left(\lambda + \frac{2}{\pi}\int_0^\pi w_x^2(x,t)\,dx\right) w_{xx}(x,t) = f(x,t), \tag{1}$$

$$0 < x < \pi, \quad 0 < t \le T,$$

with the initial boundary conditions

$$\begin{gathered} w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \\ w(0,t) = w(\pi,t) = 0, \end{gathered} \tag{2}$$

$$0 \le x \le \pi, \quad 0 \le t \le T.$$

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Here $\lambda > 0$ and T are given constants, while $f(x,t), w^0(x), w^1(x)$ are given functions.

The equation (1), when $f(x,t) = 0$, is proposed by Kirchhoff [1] in 1876. It is a generalization of D'Alembert string's oscillation model with equation $w_{tt} = c^2 w_{xx}$. Many authors researched the homogeneous equation, corresponding to (1) and its generalizations in terms of solvability. Among them are works by A. Arosio, S. Bernstein, P. D'Ancona, R. Narasimha, K. Nishihara, S. Panizzi, S. Pohozaev, S. Spagnolo and others. Some work has been done in the field of studying numerical methods. Such are works by F. Attugui, I. Christie, R. Dickey, I. Liu, V. Odisharia [3], J. Peradze [3], [4], M. Rincon, J. Rogava, J. Sanz-Serna, Z. Vashakidze and so on. The approximate methods of solution of some class of parabolic integro-differential equations is investigated by T. Jangveladze, Z. Kiguradze and B. Neta.

Here we will generalize the numerical algorithm offered in [2] for the approximate solution of problem (1), (2) for the case $f(x,t) = 0$. Then we solve test examples using this algorithm and present the results in tables and graphs.

2. Algorithm

The algorithm has three parts.
(i) As the first part is used the Galerkin method. The approximate solution is sought in the form of a finite sum $w_n(x,t) = \sum_{i=1}^n w_{ni}(t) \sin ix$. Here the coefficients $w_{ni}(t)$ are defined from the system of differential equations

$$w''_{ni}(t) + \left(\lambda + \sum_{j=1}^n j^2 w_{nj}^2(t)\right) i^2 w_{ni}(t) = f_i(t), \quad i = 1, 2, \cdots, n, \tag{3}$$

$$0 < t \le T,$$

with the conditions

$$w_{ni}(0) = a_i^0, \quad w'_{ni}(0) = a_i^1, \quad i = 1, 2, \ldots, n. \tag{4}$$

Here

$$f_i(t) = \frac{2}{\pi} \int_0^\pi f(x,t) \sin ix \, dx, \quad a_i^p = \frac{2}{\pi} \int_0^\pi w^p(x) \sin ix \, dx, \quad p = 0, 1.$$

Let us introduce the functions $u_{ni}(t) = w'_{ni}(t)$, $v_{ni}(t) = i w_{ni}(t)$, $i = 1, 2, \ldots, n$. Then system (3), (4) can be rewritten as an equivalent first-order system

$$u'_{ni}(t) + \left(\lambda + \sum_{j=1}^n v_{nj}^2(t)\right) i v_{ni}(t) = f_i(t), \tag{5}$$

$$v'_{ni}(t) = i u_{ni}(t), \quad 0 < t < T, \tag{6}$$

$$u_{ni}(0) = a_i^1, \quad v_{ni}(0) = ia_i^0, \quad i = 1, 2, \ldots, n. \tag{7}$$

(ii) As the second part is used the Crank-Nicolson type difference scheme. Problem (5)-(7) is solved by means of the difference method. On the time interval $[0, T]$ is introduced the grid $\{t_m \,|\, 0 = t_0 < t_1 < \cdots < t_M = T\}$ with in generally variable step $\tau_m = t_m - t_{m-1}$, $m = 1, 2, \ldots, M$. let us denote $f_i^m = f_i(t_m)$. Approximate values of $u_{ni}(t)$ and $v_{ni}(t)$ on the m-th layer, i. e., for $t = t_m$, $m = 1, 2, \ldots, M$, denoted by u_{ni}^m and v_{ni}^m, are defined by the implicit scheme

$$\frac{u_{ni}^m - u_{ni}^{m-1}}{\tau_m} + \left\{\lambda + \frac{1}{2}\left[\sum_{j=1}^{n}((v_{nj}^m)^2 + (v_{nj}^{m-1})^2)\right]\right\} i\frac{v_{ni}^m + v_{ni}^{m-1}}{2} = f_i^m, \tag{8}$$

$$\frac{v_{ni}^m - v_{ni}^{m-1}}{\tau_m} = i\frac{u_{ni}^m + u_{ni}^{m-1}}{2}, \quad m = 1, 2, \ldots, M, \tag{9}$$

$$u_{ni}^0 = a_i^1, \quad v_{ni}(0) = ia_i^0, \quad i = 1, 2, \ldots, n. \tag{10}$$

(iii) The third part of the algorithm is a Jacobi type iterative process. The system of nonlinear equations (8)-(10) will be solved layer by layer using iterations. Let us denote by $u_{ni}^{m,k}$ and $v_{ni}^{m,k}$, respectively, the k-th iteration approximation of u_{ni}^m and v_{ni}^m, $k = 0, 1, \ldots$. We use the following iteration method

$$u_{ni}^{m,k} = u_{ni}^{m-1,k_0} - \frac{\tau_m i}{2}\left\{\lambda + \frac{1}{2}\left[\sum_{j=1}^{n}((v_{nj}^{m,k-1})^2 + (v_{nj}^{m-1,k_0})^2)\right]\right\} \times (v_{ni}^{m,k-1} + v_{ni}^{m-1,k_0}) + \tau_m f_i^m, \tag{11}$$

$$v_{ni}^{m,k} = v_{ni}^{m-1,k_0} + \frac{\tau_m i}{2}(u_{ni}^{m,k-1} + u_{ni}^{m-1,k_0}). \tag{12}$$

Here k_0 is the number of iterations carried out on $(m-1)$-th layer.

The coefficients $u_{ni}^{m,k}$ and $v_{ni}^{m,k}$ are calculated by formulas (11), (12). Then, for $t = t_m$, an approximate value of the exact solution $w(x, t_m)$ of problem (1), (2) is written as the sum

$$w_n^{m,k}(x) = \sum_{i=1}^{n}\frac{1}{i}v_{ni}^{m,k}\sin ix.$$

Let us define error for k-th iteration by

$$\Delta_n^k = \max_m(\max_{0\le x\le\pi}\left|w(x, t_m) - w_n^{m,k}(x)\right|).$$

3. Test examples

Here we present results of calculations of two test examples.

Example 3.1 Let $T = 1$, $\lambda = 0.4$,

$$f(x,t) = 6t \sin 2x + (\lambda + 1 + 4t^6)(\sin x + 4t^3 \sin 2x),$$

$w^0(x) = \sin(x)$, $w^1(x) = 0$. The exact solution is the function $w(x,t) = \sin x + t^3 \sin 2x$. The algorithm is applied with $n = 5$, $M = 20$ and $\tau_m = 0.05$. The number of iterations is $k = 9$. The error is $\Delta_n^k = 0.0744789237$. The results are presented below in tables and graphs.

Table 1 and Table 2, respectively, contain the values of the exact and approximate solutions at chosen points. Fig. 1 presents the graph of exact solution. Fig. 2, a) and b), respectively, present slices of exact and approximate solutions graphs for $t = 0.5$ and $t = 1.0$. By wz and wm, respectively, are denoted the graps of exact and approximate solutions.

Table 1. Values of the exact solution $w(x,t)$

$t \backslash x$	0.0	$\pi/4$	$\pi/2$	$3\pi/4$	π
0.0	0.0	0.7071067812	1.00	0.7071067812	0.0
0.25	0.0	0.7227317812	1.00	0.6914817812	0.0
0.5	0.0	0.8321067812	1.00	0.5821067812	0.0
0.75	0.0	1.1289817812	1.00	0.2852317812	0.0
1.0	0.0	1.7071067812	1.00	−0.2928932188	0.0

Table 2. Values of the approximate solution $w_n^{m,k}(x)$

$t \backslash x$	0.0	$\pi/4$	$\pi/2$	$3\pi/4$	π
0.0	0.0	0.7071067812	1.0000000000	0.7071067812	0.0
0.25	0.0	0.7277079072	0.9999995195	0.6865049757	0.0
0.5	0.0	0.8513049799	0.9999784246	0.5628780703	0.0
0.75	0.0	1.1713347326	0.9997953763	0.2425894482	0.0
1.0	0.0	1.7803213024	0.9991059325	−0.3673721424	0.0

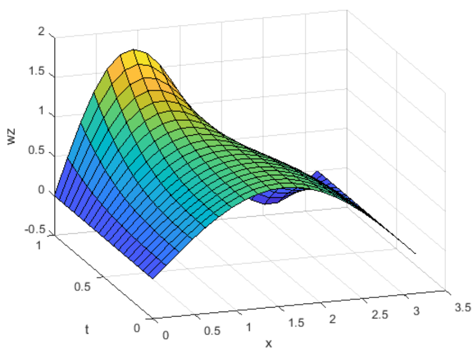

Figure 1. Graph of exact solution

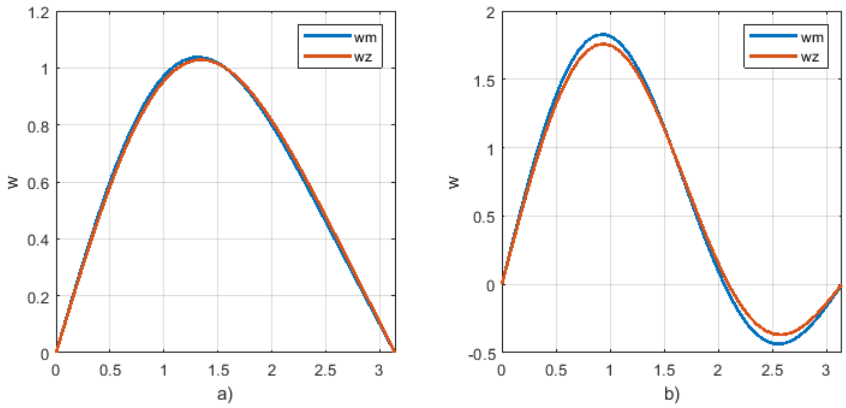

Figure 2. Slices of exact and approximate solutions graphs for $t = 0.5$ and $t = 1.0$

Example 3.2 Let $T = 1$, $\lambda = 1.0$,

$$f(x,t) = \left[\left(\frac{x}{\pi}\right)^2 \sin x - \left(\lambda + \frac{e^{2t}-1}{2t}\right)\left(\frac{2t}{\pi}\cos x - \left(1 - \left(\frac{t}{\pi}\right)^2\right)\sin x\right)\right] e^{\frac{1}{\pi}xt},$$

$w^0(x) = \sin x$, $w^1(x) = \frac{1}{\pi}x \sin x$. The exact solution is the function $w(x,t) = e^{\frac{1}{\pi}xt} \sin x$. The algorithm is applied with $n = 5, M = 20$ and $\tau_m = 0.05$. The number of iterations is $k = 10$. The error is $\Delta_n^k = 0.0441088504$. The results are presented below in tables and graphs.

Table 3 and Table 4, respectively, contain the values of the exact and approximate solutions at chosen points. Fig. 3 presents the graph of exact solution. Fig. 4, a) and b), respectively, present slices of exact and approximate solutions graphs for $t = 0.5$ and $t = 1.0$. By wz and wm, respectively, are denoted the graps of exact and approximate solutions.

Table 3. Values of the exact solution $w(x,t)$

$t\backslash x$	0.0	$\pi/4$	$\pi/2$	$3\pi/4$	π
0.0	0.0	0.7071067812	1.0000000000	0.7071067812	0.0
0.25	0.0	0.7527112504	1.1331484530	0.8529335890	0.0
0.5	0.0	0.8012569553	1.2840254167	1.0288342958	0.0
0.75	0.0	0.8529335890	1.4549914146	1.2410110493	0.0
1.0	0.0	0.9079430794	1.6487212707	1.4969450675	0.0

Table 4. Values of the approximate solution $w_n^{m,k}(x)$

$t\backslash x$	0.0	$\pi/4$	$\pi/2$	$3\pi/4$	π
0.0	0.0	0.7071067812	1.0000000000	0.7071067812	0.0
0.25	0.0	0.7517456659	1.1350376818	0.8567349797	0.0
0.5	0.0	0.7992021464	1.2918594392	1.0429980800	0.0
0.75	0.0	0.8505124399	1.4721767939	1.2706482241	0.0
1.0	0.0	0.9072624911	1.6763079963	1.5410539179	0.0

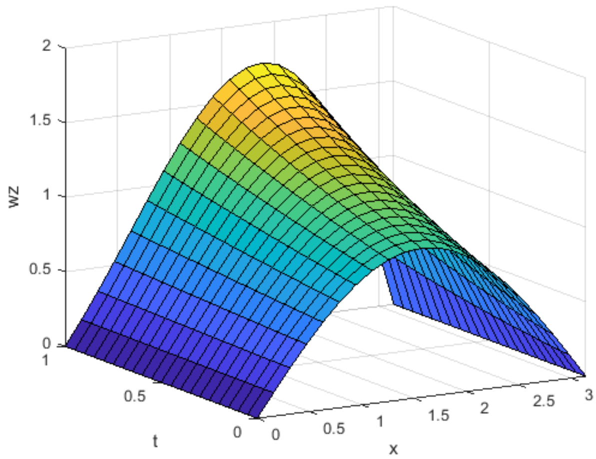

Figure 3. Graph of exact solution

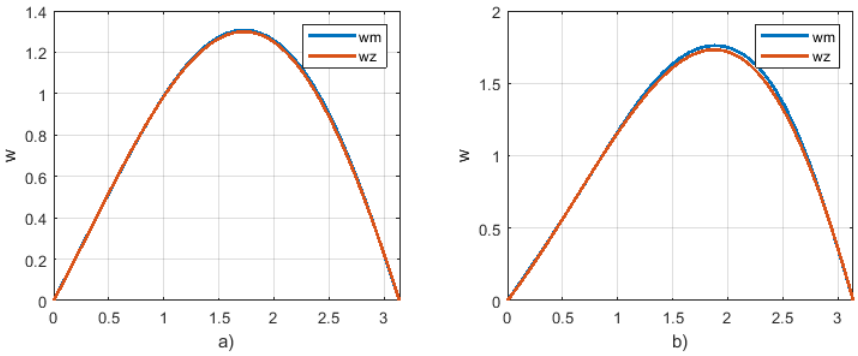

Figure 4. Slices of exact and approximate solutions graphs for $t = 0.5$ and $t = 1.0$

4. Conclusion

If we increase the values of parametres n and M, the error improves. Namely, if we take $n = 12$ and $M = 160$ in example 3.1, the error is $\Delta_n^k = 0.0093695833$. If we take $n = 12$ and $M = 80$ in example 3.2, the error is $\Delta_n^k = 0.0096361646$. Based on the obtained results, it can be concluded that the numerical algorithm for solving problem (1), (2) is effective.

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