Bulletin of TICMI
Vol. 28, No. 1, 2024, 41–50

Classes of Generalized Continuous Variation Functions on Vilenkin Groups and Their Applications

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(*Received January 25, 2024; Revised May 29, 2023; Accepted June 4, 2024*)

Classes of continuous oscillation functions for bounded Vilenkin groups are introduced in the work. Establishing embedding theorems across classes and examining the issue of summability of double Vilenkin–Fourier series.

Keywords: Vilenkin Group, Functions of bounded variation, Summability by Cesáro method.

AMS Subject Classification: 42C10, 26A45, 42B08, 43A55.

1. Introduction

The concept of classes of functions of bounded variation was introduced by Jordan in 1881 [27], and it was applied to pointwise convergence of Fourier series. Later, this concept was generalized by many authors. For example, Wiener [32], Young [33], Chanturia [15], Waterman [31], Avdispahic [5], Belov [11] Kita and Yoneda [28], Akhobadze [1-3], Berezhnoi [12-14]. The following were studied in these classes:

- Pointwise convergence of Fourier series;
- Uniform convergence of Fourier series;
- Absolute convergence of Fourier series;
- Summability of Fourier series;
- Rate of the convergence of Fourier series (approximation properties);
- Embedding theorems.

The space of functions of bounded variation for functions of two variables was introduced by Hardy in 1906 [26]. This class has been generalized in the paper of Golubov [25], Sahakian [29], Dyachenko, Waterman [16], Bakhvalov [7-10] and Goginava, Sahakian [17-23].

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ISSN: 1512-0082 print

The classes of bounded oscillation functions and their generalisations to Walsh and Vilenkin groups G_m were investigated in the following publications [4, 6, 30].

Our goal in the presented work is to introduce $C\Lambda_1\Lambda_2 O\left(G_m^2\right)$ and $C\Lambda^{\#}O\left(G_m^2\right)$ classes of continuous oscillation functions on Vilenkin groups $G_m^2 := G_m \times G_m$. The embedding theorem between classes is used to solve the summability problem of double Vilenkin-Fourier series.

2. Bounded Vilenkin group

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, ...)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, ..., m_k - 1\}$ the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the groups Z_{m_j}, with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures $\mu_k(\{j\}) := \frac{1}{m_k}$ $(j \in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m) = 1$. If the sequence m is bounded, then G_m is called a bounded Vilenkin group. The elements of G_m can be represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, $(x_j \in Z_{m_j})$. The group operation $+$ in G_m is given by $x + y = (x_0 + y_0 \,(\mathrm{mod} m_0), ..., x_k + y_k \,(\mathrm{mod} m_k), ...)$, where $x = (x_0, ..., x_k, ...)$ and $y = (y_0, ..., y_k, ...) \in G_m$. The inverse of $+$ will be denoted by $-$. In this paper we will consider only bounded Vilenkin group. Set $e_n := (0, ..., 0, 1, 0, ...) \in G_m$ the n th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$ .If we define the so-called generalized number system based on m in the following way: $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} \varepsilon_j(n) M_j$, where $\varepsilon_j(n) \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of $\varepsilon_j(n)$'s differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions in this way

$$r_k(x) := \exp\left(\frac{2\pi \imath x_k}{m_k}\right) \;(\imath^2 = -1,\; x \in G_m,\; k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

We consider the double system $\{\psi_k(x) \times \psi_l(y) : \; k, l \in \mathbb{N}\}$ on $G_m^2 := G_m \times G_m$.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series,the Dirichlet kernels with respect to the two-dimensional Vilenkin system are defined as follow:

$$\widehat{f}(n_1, n_2) := \int_{G_m^2} f(x, y)\, \bar{\psi}_{n_1}(x)\, \bar{\psi}_{n_2}(y)\, d\mu(x, y),$$

$$S_{n_1,n_2}(x,y,f) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\widehat{f}(k_1,k_2)\,\psi_{k_1}(x)\,\psi_{k_2}(y)\,,$$

$$D_{n_1,n_2}(x,y) := D_{n_1}(x)\,D_{n_2}(y)\,,$$

The $(C,-\alpha_1,-\alpha_2)$ means of the two-dimensional Vilenkin-Fourier series are defined as

$$\sigma_{n_1,n_2}^{-\alpha_1,-\alpha_2}(x,y,f) = \frac{1}{A_{n_1}^{-\alpha_1}A_{n_2}^{-\alpha_2}}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}A_{n_1-i}^{-\alpha_1}A_{n_2-j}^{-\alpha_2}\hat{f}(i,j)\psi_i(u)\,\psi_j(v)\,.$$

3. Generalized continuous oscilation on Vilenkin group

Now let us introduce the concepts of generalized variations on the group G_m^2. We assume that,

$$1 < \lambda_1^s \le \lambda_2^s \le \ldots, \qquad \sum_{n=1}^{\infty}\frac{1}{\lambda_n^s} = \infty\,(s=1,2)\,.$$

Set

$$Z_\beta^{(k)} = (x_0,...,x_{k-1},0,...)\,,$$

where

$$\beta := \sum_{j=0}^{k-1}\left(\frac{x_j}{M_{j+1}}\right)M_k\left(x_j \in Z_{m_j}\right).$$

Define

$$\operatorname{osc}_1\left(f, Z_{\pi_1(\beta_1)}^{(k_1)} + I_{k_1}, y\right) := \sup_{x,x' \in Z_{\pi_1(\beta_1)}^{(k_1)} + I_{k_1}}\left|f(x,y) - f\left(x',y\right)\right|,$$

$$\operatorname{osc}_2\left(f, x, Z_{\pi_2(\beta_2)}^{(k_2)} + I_{k_2}\right) := \sup_{y,y' \in Z_{\pi_2(\beta_2)}^{(k_2)} + I_{k_2}}\left|f(x,y) - f\left(x,y'\right)\right|$$

and

$$\operatorname{osc}_{1,2}\left(f, Z_{\pi(\beta_1)}^{(k_1)} + I_{k_1}, Z_{\pi(\beta_2)}^{(k_2)} + I_{k_2}\right)$$
$$: = \sup_{y,y' \in Z_{\pi_2(\beta_2)}^{(k_2)} + I_{k_2},\, x,x' \in Z_{\pi_1(\beta_1)}^{(k_1)} + I_{k_1}}\left|f(x,y) - f\left(x,y'\right) - f\left(x',y\right) + f\left(x',y'\right)\right|,$$

where π_1 and π_2 are permutations of the sets $\{0,1,...,M_{k_1}-1\}$ and $\{0,1,...,M_{k_2}-1\}$, respectively. For the sequence of positive numbers $\Lambda^1 := \left\{\lambda_n^1 : n \in \mathbb{N}\right\}$ and $\Lambda^2 := \left\{\lambda_n^2 : n \in \mathbb{N}\right\}$ we denote

$$\Lambda^1 O_1\left(f;G_m^2\right)=\sup_y\sup_{k_1}\sup_{\pi_1}\sum_{\beta_1=0}^{M_{k_1}-1}\frac{\operatorname{osc}_1\left(f,Z^{(k_1)}_{\pi_1(\beta_1)}+I_{k_1},y\right)}{\lambda^1_{\beta_1}},$$

$$\Lambda^2 O_2\left(f;G_m^2\right)=\sup_x\sup_{k_2}\sup_{\pi_2}\sum_{\beta_2=0}^{M_{k_2}-1}\frac{\operatorname{osc}_2\left(f,x,Z^{(k_2)}_{\pi_2(\beta_2)}+I_{k_2}\right)}{\lambda^2_{\beta_2}}$$

and

$$\Lambda^1\Lambda^2 O_{1,2}\left(f;G_m^2\right)=\sup_{k_1,k_2}\sup_{\pi_1,\pi_2}\sum_{\beta_1=0}^{M_{k_1}-1}\sum_{\beta_2=0}^{M_{k_2}-1}\frac{\operatorname{osc}_{1,2}\left(f,Z^{(k_1)}_{\pi_1(\beta_1)}+I_{k_1},Z^{(k_2)}_{\pi_2(\beta_2)}+I_{k_2}\right)}{\lambda^1_{\beta_1}\lambda^2_{\beta_2}}.$$

Definition 3.1: Let $\Lambda^s=\{\lambda_n^s\}_{n=1}^{\infty}$ and $\Lambda_k^s=\{\lambda_n^s\}_{n=k}^{\infty}$, $s=1,2,\ k=1,2,\ldots$. We say that the function f is continuous in $\Lambda_1\Lambda_2$-oscilation and write $f\in C\Lambda_1\Lambda_2 O\left(G_m^2\right)$, if

$$\lim_{k\to\infty}\Lambda_k^1 O_1\left(f;G_m^2\right)=0,$$

$$\lim_{k\to\infty}\Lambda_k^2 O_2\left(f;G_m^2\right)=0,$$

$$\lim_{k\to\infty}\Lambda_k^1\Lambda^2 O_{1,2}\left(f;G_m^2\right)=0$$

and

$$\lim_{k\to\infty}\Lambda^1\Lambda_k^2 O_{1,2}\left(f;G_m^2\right)=0.$$

Define

$$\Lambda^{\#} O_1\left(f;G_m^2\right)=\sup_{k_1}\sup_{\pi_1}\sup_{\{y_{\beta_1}\}}\sum_{\beta_1=0}^{M_{k_1}-1}\frac{\operatorname{osc}_1\left(f,Z^{(k_1)}_{\pi_1(\beta_1)}+I_{k_1},y_{\beta_1}\right)}{\lambda^1_{\beta_1}}$$

and

$$\Lambda^{\#} O_2\left(f;G_m^2\right)=\sup_{k_2}\sup_{\pi_2}\sup_{\{x_{\beta_2}\}}\sum_{\beta_2=0}^{M_{k_2}-1}\frac{\operatorname{osc}_2\left(f,x_{\beta_2},Z^{(k_2)}_{\pi_2(\beta_2)}+I_{k_2}\right)}{\lambda^2_{\beta_2}}.$$

Definition 3.2: Let $\Lambda^s=\{\lambda_n^s\}_{n=1}^{\infty}$ and $\Lambda_k^s=\{\lambda_n^s\}_{n=k}^{\infty}$, $s=1,2,\ k=1,2,\ldots$. We say that the function f is continuous in $\Lambda^{\#}$-oscilation and write $f\in C\Lambda^{\#} O\left(G_m^2\right)$, if

$$\lim_{k\to\infty}\Lambda_k^{\#} O_s\left(f;G_m^2\right)=0\quad(s=1,2).$$

Throughout the paper, instead of the following inequality $a<c\cdot b$, the notation $a\lesssim b$ will be used, where the constant c may depend on α_1 and α_2.

4. Embedding theorem

Theorem 4.1 : *Let* $\alpha_1, \alpha_2 \in (0,1)$ *and* $\alpha_1 + \alpha_2 < 1$*. Then*

$$C\left\{i^{1-(\alpha_1+\alpha_2)}\right\}^{\#} O\left(G_m^2\right) \subset C\left\{i^{1-\alpha_1}\right\}\left\{j^{1-\alpha_2}\right\} O\left(G_m^2\right).$$

Proof : [Proof of Theorem 4.1] We assume that $f \in C\left\{i^{1-(\alpha_1+\alpha_2)}\right\}^{\#} O\left(G_m^2\right)$ and must demonstrate that the conditions of definitions 3.1 are satisfied. It is easy to see that

$$\left\{i^{1-\alpha_1}\right\}_n O_1\left(f; G_m^2\right) \leq \left\{i^{1-(\alpha_1+\alpha_2)}\right\}_n^{\#} O_1\left(f; G_m^2\right) \to 0 \tag{1}$$

as $n \to \infty$. Analogously, we can prove that

$$\left\{j^{1-\alpha_2}\right\}_n O_2\left(f; G_m^2\right) \to 0 \tag{2}$$

as $n \to \infty$. Now we prove that $\lim_{k\to\infty}\left\{i^{1-\alpha_1}\right\}_k\left\{j^{1-\alpha_2}\right\} O_{1,2}\left(f; G_m^2\right) = 0$. According to the Supremum's definition, it is sufficient to demonstrate that

$$\lim_{n\to\infty} \sum_{\beta_1=n}^{M_{k_1}-1} \sum_{\beta_2=0}^{M_{k_2}-1} \frac{\left|f\left(x'_{\beta_1}, y'_{\beta_2}\right) - f\left(x_{\beta_1}, y'_{\beta_2}\right) - f\left(x'_{\beta_1}, y_{\beta_2}\right) + f\left(x_{\beta_1}, y_{\beta_2}\right)\right|}{\beta_1^{1-\alpha_1}\beta_2^{1-\alpha_2}} = 0.$$

where

$$\left(x'_{\beta_1}, y'_{\beta_2}\right), \left(x_{\beta_1}, y_{\beta_2}\right) \in \left(Z_{\pi_1(\beta_1)}^{(k_1)} + I_{k_1}\right) \times \left(Z_{\pi_2(\beta_2)}^{(k_2)} + I_{k_2}\right).$$

First, consider the case when $M_{k_1} \geq M_{k_2}$. We can write

$$\begin{aligned} &\sum_{\beta_1=n}^{M_{k_1}-1} \sum_{\beta_2=0}^{M_{k_2}-1} \frac{\left|f\left(x'_{\beta_1}, y'_{\beta_2}\right) - f\left(x_{\beta_1}, y'_{\beta_2}\right) - f\left(x'_{\beta_1}, y_{\beta_2}\right) + f\left(x_{\beta_1}, y_{\beta_2}\right)\right|}{\beta_1^{1-\alpha_1}\beta_2^{1-\alpha_2}} \qquad (3) \\ &= \left(\sum_{\beta_1=n}^{M_{k_2}-1} \sum_{\beta_2=0}^{\beta_1-1} + \sum_{\beta_1=n}^{M_{k_2}-1} \sum_{\beta_2=\beta_1+1}^{M_{k_2}-1} + \sum_{\beta_1=M_{k_2}}^{M_{k_1}-1} \sum_{\beta_2=0}^{M_{k_2}-1}\right) \\ &\quad \frac{\left|f\left(x'_{\beta_1}, y'_{\beta_2}\right) - f\left(x_{\beta_1}, y'_{\beta_2}\right) - f\left(x'_{\beta_1}, y_{\beta_2}\right) + f\left(x_{\beta_1}, y_{\beta_2}\right)\right|}{\beta_1^{1-\alpha_1}\beta_2^{1-\alpha_2}} \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

We have

$$A_1 \leq \sum_{\beta_1=n}^{M_{k_2}-1} \frac{1}{\beta_1^{1-\alpha_1}} \sum_{\beta_2=0}^{\beta_1-1} \frac{\left|f\left(x'_{\beta_1}, y'_{\beta_2}\right) - f\left(x_{\beta_1}, y'_{\beta_2}\right)\right|}{\beta_2^{1-\alpha_2}} \tag{4}$$

$$+ \sum_{\beta_1=n}^{M_{k_2}-1} \frac{1}{\beta_1^{1-\alpha_1}} \sum_{\beta_2=0}^{\beta_1-1} \frac{\left|f\left(x_{\beta_1}, y_{\beta_2}\right) - f\left(x'_{\beta_1}, y_{\beta_2}\right)\right|}{\beta_2^{1-\alpha_2}}$$

$$\lesssim \sum_{\beta_1=n}^{M_{k_2}-1} \frac{\left|f\left(x'_{\beta_1}, \overline{y}'_{\beta_1}\right) - f\left(x_{\beta_1}, \overline{y}'_{\beta_1}\right)\right|}{\beta_1^{1-\alpha_1-\alpha_2}}$$

$$\lesssim \left\{i^{1-\alpha_1-\alpha_2}\right\}_n^{\#} O_1\left(f; G_m^2\right) \to 0$$

as $n \to \infty$, where

$$\left|f\left(x'_{\beta_1}, \overline{y}'_{\beta_1}\right) - f\left(x_{\beta_1}, \overline{y}'_{\beta_1}\right)\right| = \sup_{0 \leq \beta_2 < \beta_1} \left|f\left(x_{\beta_1}, y_{\beta_2}\right) - f\left(x'_{\beta_1}, y_{\beta_2}\right)\right|.$$

Analogously, we have

$$A_2 \lesssim \sum_{\beta_2=n}^{M_{k_2}-1} \frac{\left|f\left(\overline{x}_{\beta_2}, y'_{\beta_2}\right) - f\left(\overline{x}_{\beta_2}, y_{\beta_2}\right)\right|}{\beta_2^{1-\alpha_1-\alpha_2}} \tag{5}$$

$$\lesssim \left\{i^{1-\alpha_1-\alpha_2}\right\}_n^{\#} O_2\left(f; G_m^2\right) \to 0$$

as $n \to \infty$, where

$$\left|f\left(\overline{x}_{\beta_2}, y'_{\beta_2}\right) - f\left(\overline{x}_{\beta_2}, y_{\beta_2}\right)\right| = \sup_{0 \leq \beta_1 < \beta_2} \left|f\left(x_{\beta_1}, y'_{\beta_2}\right) - f\left(x_{\beta_1}, y_{\beta_2}\right)\right|.$$

Using the same approach, we get

$$A_3 \leq \sum_{\beta_1=M_{k_2}}^{M_{k_1}-1} \frac{\left|f\left(x'_{\beta_1}, \overline{y}_{\beta_1}\right) - f\left(x_{\beta_1}, \overline{y}_{\beta_1}\right)\right|}{\beta_1^{1-\alpha_1}} \sum_{\beta_2=0}^{M_{k_2}-1} \frac{1}{\beta_2^{1-\alpha_2}} \tag{6}$$

$$\lesssim \sum_{\beta_1=M_{k_2}}^{M_{k_1}-1} \frac{\left|f\left(x'_{\beta_1}, \overline{y}_{\beta_1}\right) - f\left(x_{\beta_1}, \overline{y}_{\beta_1}\right)\right|}{\beta_1^{1-\alpha_1-\alpha_2}}$$

$$\lesssim \left\{i^{1-\alpha_1-\alpha_2}\right\}_n^{\#} O_1\left(f; G_m^2\right) \to 0$$

as $n \to \infty$. Now consider the case when $M_{k_1} < M_{k_2}$ and we can write

$$\sum_{\beta_1=n}^{M_{k_1}-1}\sum_{\beta_2=0}^{M_{k_2}-1}\frac{\left|f\left(x'_{\beta_1},y'_{\beta_2}\right)-f\left(x_{\beta_1},y'_{\beta_2}\right)-f\left(x'_{\beta_1},y_{\beta_2}\right)+f\left(x_{\beta_1},y_{\beta_2}\right)\right|}{\beta_1^{1-\alpha_1}\beta_2^{1-\alpha_2}} \tag{7}$$

$$=\underbrace{\sum_{\beta_1=n}^{M_{k_1}-1}\sum_{\beta_2=0}^{\beta_1}\frac{\left|f\left(x'_{\beta_1},y'_{\beta_2}\right)-f\left(x_{\beta_1},y'_{\beta_2}\right)-f\left(x'_{\beta_1},y_{\beta_2}\right)+f\left(x_{\beta_1},y_{\beta_2}\right)\right|}{\beta_1^{1-\alpha_1}\beta_2^{1-\alpha_2}}}_{B_1}$$

$$+\underbrace{\sum_{\beta_1=n}^{M_{k_1}-1}\sum_{\beta_2=\beta_1+1}^{M_{k_2}-1}\frac{\left|f\left(x'_{\beta_1},y'_{\beta_2}\right)-f\left(x_{\beta_1},y'_{\beta_2}\right)-f\left(x'_{\beta_1},y_{\beta_2}\right)+f\left(x_{\beta_1},y_{\beta_2}\right)\right|}{\beta_1^{1-\alpha_1}\beta_2^{1-\alpha_2}}}_{B_2}.$$

As above we obtain the following estimates

$$B_1 \lesssim \left\{i^{1-\alpha_1-\alpha_2}\right\}_n^{\#} O_1\left(f;G_m^2\right)\to 0 \tag{8}$$

and

$$B_2 \lesssim \left\{i^{1-\alpha_1-\alpha_2}\right\}_n^{\#} O_2\left(f;G_m^2\right)\to 0 \tag{9}$$

as $n\to\infty$.

Combining (1)-(9) we complete the proof of Theorem 4.1. □

5. Summability of double Vilenkin-Fourier series

Set

$$\Delta_2 f\left(x,y,e_{k_2}\right):=f\left(x,y-e_{k_2}\right)-f\left(x,y\right),$$
$$\Delta_1 f\left(x,y,e_{k_1}\right):=f\left(x-e_{k_1},y\right)-f\left(x,y\right)$$

and

$$\Delta_{12} f\left(x,y,e_{k_1},e_{k_2}\right):=f\left(x-e_{k_1},y-e_{k_2}\right)-f\left(x-e_{k_1},y\right)-f\left(x,y-e_{k_2}\right)+f\left(x,y\right).$$

Assume $C\left(G_m^2\right)$ is the space of continuous functions defined on group G_m^2 with supreme norm (C-norm). The dyadic partial moduli of continuity of a function $f\in C\left(G_m^2\right)$ in the C-norm are defined by

$$\omega_1\left(f,\frac{1}{M_n}\right)_C=\sup_{u\in I_n}\left\|f\left(\cdot-u,\cdot\right)-f\left(\cdot,\cdot\right)\right\|_C,$$
$$\omega_2\left(f,\frac{1}{M_n}\right)_C=\sup_{v\in I_n}\left\|f\left(\cdot,\cdot-v\right)-f\left(\cdot,\cdot\right)\right\|_C,$$

while the dyadic mixed modulus of continuity is defined as follows:

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)_C$$
$$=\sup_{(u,v)\in I_n\times I_m}\|f(\cdot-u,\cdot-v)-f(\cdot-u,\cdot)-f(\cdot,\cdot-v)+f(\cdot,\cdot)\|_C.$$

Let us now formulate the central theorem.

Theorem 5.1: *Let* $f\in C\left(G_m^2\right)\cap C\left\{i^{1-\alpha_1}\right\}\left\{i^{1-\alpha_2}\right\}O\left(G_m^2\right)$ *and* $\alpha_1,\alpha_2\in(0,1)$. *Then*

$$\lim_{\min(n_1,n_2)\to\infty}\left\|\sigma_{n_1,n_2}^{-\alpha_1,-\alpha_2}(f,\cdot,\cdot)-f(\cdot,\cdot)\right\|_{C(G_m^2)}=0.$$

Proof: [Proof of Theorem 5.1] To establish the theorem, just test the validity of the following three conditions (see [24])

$$\lim_{\min(n_1,n_2)\to\infty}\left\|\sum_{\beta_1=1}^{M_{k_1}-1}\sum_{\beta_2=1}^{M_{k_2}-1}\frac{1}{\beta_1^{1-\alpha_1}}\frac{1}{\beta_2^{1-\alpha_2}}\left|\Delta_{1,2}\left(\cdot-Z_{\beta_1}^{(k_1)},\cdot-Z_{\beta_2}^{(k_2)}\right)\right|\right\|_{C(G_m^2)}=0, \quad (10)$$

$$\lim_{n_2\to\infty}\left\|\sum_{\beta_2=1}^{M_{k_2}-1}\frac{1}{\beta_2^{1-\alpha_2}}\left|\Delta_2\left(\cdot,\cdot-Z_{\beta_2}^{(k_2)}\right)\right|\right\|_{C(G_m^2)}=0 \quad (11)$$

and

$$\lim_{n_1\to\infty}\left\|\sum_{\beta_1=1}^{M_{k_1}-1}\frac{1}{\beta_1^{1-\alpha_1}}\left|\Delta_1\left(\cdot-Z_{\beta_1}^{(k_1)},\cdot\right)\right|\right\|_{C(G_m^2)}=0. \quad (12)$$

Let $\{\theta_1(M_{k_1})\}$, $\{\theta_2(M_{k_2})\}$ and $\theta_3(M_{k_1},M_{k_2})$ be a subsequence of natural numbers which satisfie the following conditions:

$$\theta_1(M_{k_1}),\theta_2(M_{k_2}),\theta_3(M_{k_1},M_{k_2})\to\infty,$$

$$\omega_1\left(f,\frac{1}{M_{k_1}}\right)_C\theta_1^{\alpha_1}(M_{k_1})\to 0, \quad (13)$$

$$\omega_2\left(f,\frac{1}{M_{k_2}}\right)_C\theta_2^{\alpha_2}(M_{k_2})\to 0 \quad (14)$$

and

$$\omega_{1,2}\left(f,\frac{1}{M_{k_1}},\frac{1}{M_{k_2}}\right)_C\theta_3^{\alpha_1+\alpha_2}(M_{k_1},M_{k_2})\to 0. \quad (15)$$

as $k_1, k_2 \to \infty$. Firstly, let us verify that due to (13) the condition (11) is true. In a similar manner, using (14) the fairness of condition (12) may be verified

$$\sum_{\beta_2=1}^{M_{k_2}-1} \frac{1}{\beta_2^{1-\alpha_2}} \left|\Delta_2\left(x, y - Z_{\beta_2}^{(k_2)}\right)\right|$$

$$= \sum_{\beta_2=1}^{\theta_2\left(M_{k_2}\right)-1} \frac{1}{\beta_2^{1-\alpha_2}} \left|\Delta_2\left(x, y - Z_{\beta_2}^{(k_2)}\right)\right| + \sum_{\beta_2=\theta_2\left(M_{k_2}\right)}^{M_{k_2}-1} \frac{1}{\beta_2^{1-\alpha_2}} \left|\Delta_2\left(x, y - Z_{\beta_2}^{(k_2)}\right)\right|$$

$$\lesssim \left\{\omega_2\left(f, \frac{1}{M_n}\right)_C \theta^{\alpha_2}\left(M_{k_2}\right) + \left\{i^{1-\alpha_2}\right\}_{\theta\left(M_{k_2}\right)} O_2\left(f; G_m^2\right)\right\} \to 0$$

as $n_2 \to \infty$. Analogously, we can prove (10) using (15). Indeed, we have

$$\sum_{\beta_1=1}^{M_{k_1}-1} \sum_{\beta_2=1}^{M_{k_2}-1} \frac{1}{\beta_1^{1-\alpha_1}} \frac{1}{\beta_2^{1-\alpha_2}} \left|\Delta_{1,2}\left(x - Z_{\beta_1}^{(k_1)}, y - Z_{\beta_2}^{(k_2)}\right)\right|$$

$$\leq \sum_{\beta_1=1}^{\theta_3\left(M_{k_1}, M_{k_2}\right)-1} \sum_{\beta_2=1}^{\theta_3\left(M_{k_1}, M_{k_2}\right)-1} \frac{1}{\beta_1^{1-\alpha_1}} \frac{1}{\beta_2^{1-\alpha_2}} \left|\Delta_{1,2}\left(x - Z_{\beta_1}^{(k_1)}, y - Z_{\beta_2}^{(k_2)}\right)\right|$$

$$+ \sum_{\beta_1=1}^{M_{k_1}-1} \sum_{\beta_2=\theta_3\left(M_{k_1}, M_{k_2}\right)}^{M_{k_2}-1} \frac{1}{\beta_1^{1-\alpha_1}} \frac{1}{\beta_2^{1-\alpha_2}} \left|\Delta_{1,2}\left(x - Z_{\beta_1}^{(k_1)}, y - Z_{\beta_2}^{(k_2)}\right)\right|$$

$$+ \sum_{\beta_1=\theta_3\left(M_{k_1}, M_{k_2}\right)}^{M_{k_1}-1} \sum_{\beta_2=1}^{M_{k_2}-1} \frac{1}{\beta_1^{1-\alpha_1}} \frac{1}{\beta_2^{1-\alpha_2}} \left|\Delta_{1,2}\left(x - Z_{\beta_1}^{(k_1)}, y - Z_{\beta_2}^{(k_2)}\right)\right|$$

$$\lesssim \left\{\omega_{1,2}\left(f, \frac{1}{M_{k_1}}, \frac{1}{M_{k_2}}\right)_C \theta_3^{\alpha_1+\alpha_2}\left(M_{k_1}, M_{k_2}\right)\right.$$

$$+ \left\{i^{1-\alpha_1}\right\}\left\{j^{1-\alpha_2}\right\}_{\theta_3\left(M_{k_1}, M_{k_2}\right)} O_{1,2}\left(f; G_m^2\right)$$

$$\left. + \left\{i^{1-\alpha_1}\right\}_{\theta_3\left(M_{k_1}, M_{k_2}\right)}\left\{j^{1-\alpha_2}\right\} O_{1,2}\left(f; G_m^2\right)\right\}.$$

The following is obtained from Theorem 4.1 and Theorem 5.1 □

Theorem 5.2: *Let* $f \in C\left(G_m^2\right) \cap C\left\{i^{1-(\alpha_1+\alpha_2)}\right\}^{\#} O\left(G_m^2\right)$ *and* $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_1 + \alpha_2 < 1$. *Then*

$$\lim_{\min(n_1,n_2)\to\infty} \left\|\sigma_{n_1,n_2}^{-\alpha_1,-\alpha_2}(f, \cdot, \cdot) - f(\cdot, \cdot)\right\|_{C(G_m^2)} = 0.$$

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