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On the Solution of Dirichlet Generalized and Classical Spatial Harmonic Problems by the MPS in Neighborhood of the Considered Domain Surface

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The Dirichlet generalized and classical harmonic problems for the special type irregular 3-sided pyramidal domain is considered. Under a generalized problem is meant the problem when a boundary function has a finite number of first kind discontinuity curves. In the considered case, edges of the pyramid are in a role of the mentioned curves and height of the pyramid passes through the vertex of acute angle of base. In spite of difficulty of the problem domain, the algorithm for numerical solution of the boundary problem is constructed, which consists of the following main steps: a) application of the method of probabilistic solution (MPS), which in its turn is based on a computer modeling of the Wiener process; b) finding the intersection point of the path of Wiener process simulation and the pyramid surface; c) development of a code for the numerical realization and checking the accuracy of calculated results; d) calculating the meaning of a sought for function at any chosen points, which lie in the neighborhood of the domain surface. For illustration, numerical an example is considered and results are presented.

Keywords: Dirichlet generalized and classical harmonic problems, Method of probabilistic solution, Wiener process, Pyramidal domain.

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1. Introduction

It is known(see e.g., [1-5]) that in practical stationary problems (for example, determination of electrical potential, temperature potential, gravitational potential, and so on) there are cases when it is necessary to consider the Dirichlet generalized harmonic problem.

As is well known(see e.g., [1,6]) the methods used to obtain an approximate solution to classical boundary-value problems are: a)less suitable or b)useless for solving generalized boundary problems. In the first case, convergence of the approximate process is very slow in the neighborhood of discontinuity curves and, consequently, the accuracy of approximate solution of the generalized problem is

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very low(see,e.g., [1-5]). In the second case, the process is unstable. For example, a similar phenomenon takes place when solving the 3D Dirichlet generalized harmonic problem by the MFS.

Therefore researchers have tried to conduct preliminary "improvements" of the boundary value problem in question. For the Dirichlet generalized plane harmonic problems the following methods were elaborated:I) A method of reduction of the Dirichlet generalized harmonic problem to a classical problem (see, e.g., [7,8]); II) A method of conformal mapping (see, e.g., [9]); III) A method of probabilistic solution(see, e.g., [10,11]). It is evident, that in the case of 3D Dirichlet harmonic problems, from the above mentioned methods we can apply only the third one.

For 3D Dirichlet generalized harmonic problems researchers face more significant difficulties. In particular, there does not exist a universal approach that can be applied to a wide class of domains.

The above-mentioned literature [1-5] deals with the simplified generalized problems. Mainly, the methods of separation of variables, particular solutions and heuristic methods are applied to their solution. Respectively, the accuracy of the solution is low. The heuristic methods do not guarantee to find the best solution. Moreover, in some cases, they may give an incorrect solution and, thus we have to check solutions in order to establish how they satisfy all conditions of a problem(see,e.g., [1]). Therefore, the construction of effective computational schemes with a high accuracy for numerical solution of 3D Dirichlet generalized harmonic problems, applicable to a wide class of domains, are both of theoretical and practical importance.

It should be noted that in [4], the existence of discontinuity curves is ignored while solving the Dirichlet generalized harmonic simplest problems for a sphere. This fact and also the application of classical methods is the main reason of low accuracy. Therefore the for numerical solution of 3D Dirichlet generalized harmonic problems we should apply such methods which do not require approximation of a boundary function and in which the existence of discontinuity curves is not ignored. The MPS is one such method.

2. Mathematical formulation of the generalized problem

Let D be the interior of an irregular 3-sided pyramid $P_3(h) \equiv P_3$ in the space R^3, where h is its height. According to the above-mentioned we consider the case, when h coincides with the lateral edge of P_3 and passes through the vertex of acute angle of base. Without loss of generality, we assume that h is located on Ox_3 of the right-handed Cartesian coordinate system $Ox_1x_2x_3$ and the base of P_3 lies in the plane Ox_1x_2. Besides, we assume that the vertices A_1, A_2, A_3 of the base of P_3 are located in a counter-clockwise direction. Let us formulate the following problem for the pyramid $P_3 \equiv \overline{D}$.

Problem A. The function $g(y)$, given on the boundary S of the pyramid P_3 is continuous everywhere, except edges $l_1, l_2, ..., l_6$, of P_3, which represent the first kind discontinuity curves for the function $g(y)$. It is required to find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D} \setminus \bigcup\limits_{k=1}^{6} l_k)$, satisfying the following conditions:

$$\Delta u(x) = 0, \qquad x \in D, \tag{2.1}$$

$$u(y) = g(y), \qquad y \in S, \quad y \,\overline{\in}\, l_k \subset S \;\; (k = \overline{1,6}), \tag{2.2}$$

$$|u(x)| < c, \qquad\qquad x \in \overline{D}, \tag{2.3}$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $l_k (k = \overline{1,6})$ are edges of P_3, and c is a real constant.

It is shown (see [12,13]) that Problem (2.1) -(2.3) has a unique solution depending continuously on the initial data. For the generalized solution $u(x)$, the generalized extremum principle is valid,

$$\min_{x \in S} u(x) < \underset{x \in D}{u(x)} < \max_{x \in S} u(x), \tag{2.4}$$

where it is supposed that $x \overline{\in} l_k (k = \overline{1,6})$ for $x \in S$.

Note (see [12]) that the additional requirement (2.3) of the boundedness plays an important role in the extremum principle (2.4); it concerns only the neighborhoods of discontinuity curves of the function $g(y)$.

On the basis of (2.3), the values of $u(y)$ are, in general, not uniquely defined on the curves l_k. In particular, if Problem A concerns the determination of a thermal (or electric) field, then $u(y) = 0$ when $y \in l_k$, respectively. In this case, in the physical sense, the curves l_k are non-conductors (or dielectrics). Otherwise, l_k will not be a discontinuity curve.

It is evident that, in the above-mentioned case, the boundary function $g(y)$ has the following form

$$g(y) = \begin{cases} g_1(y), & y \in S_1, \\ g_2(y), & y \in S_2, \\ g_3(y), & y \in S_3, \\ g_4(y), & y \in S_4, \\ 0, & y \in l_k \; (k = \overline{1,6}), \end{cases} \tag{2.5}$$

where: S_i $(i = \overline{1,3})$ and S_4 are the lateral faces and the base of P_3 out of boundaries), respectively; the functions $g_i(y)$, $y \in S_i$ $(i = \overline{1,4})$ are continuous on the parts S_i of S. It is evident that $S = (\bigcup_{i=1}^{4} S_i) \bigcup (\bigcup_{k=1}^{6} l_k)$.

Remark 1 : a) If the interior of S is empty then we have the generalized problem with respect to closed shells.

3. The method of probabilistic solution and simulation of the Wiener process

This section briefly describes the proposed algorithm for numerical solving the problems of type A. Its detailed description is suggested in [14]. The main theorem that allows us to apply the MPS is the following one (see e.g., [13]).

Theorem 3.1 : *If a finite domain $D \in R^3$ is bounded by a piece-wise smooth surface S and $g(y)$ is continuous(or discontinuous) bounded function on S, then the solution of the Dirichlet classical (or generalized) boundary value problem for the Laplace equation at the fixed point $x \in D$ has the following form*

$$u(x) = E_x g(x(\tau)). \tag{3.1}$$

In (3.1): $E_x g(x(\tau))$ is the mathematical expectation of values of the boundary function $g(y)$ at the random intersection points of the continuous Wiener process trajectory and the boundary S; τ is a random moment of first exit of the Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If the number N of the random intersection points $y^j = (y_1^j, y_2^j, y_3^j) \in S \quad (j = 1, 2, \cdots, N)$ is sufficiently large, then according to the law of large numbers, from (3.1) we have

$$u(x) \approx u_N(x) = \frac{1}{N}\sum_{j=1}^{N} g(y^j) \tag{3.2}$$

or $u(x) = \lim u_N(x)$ for $N \to \infty$, in a probability sense. Thus, if we have the Wiener process, then the approximate value of the probabilistic solution to the Problem A at a point $x \in D$ is calculated by formula (3.2).

In order to simulate the Wiener process, we construct the following recursion (see e.g., [14]):

$$\begin{gathered} x_1(t_k) = x_1(t_{k-1}) + \gamma_1(t_k)/nq, \\ x_2(t_k) = x_2(t_{k-1}) + \gamma_2(t_k)/nq, \\ x_3(t_k) = x_3(t_{k-1}) + \gamma_3(t_k)/nq, \\ (k = 1, 2, \cdots),\ x(t_0) = x, \end{gathered} \tag{3.3}$$

according to which the coordinates of the point $x(t_k) = (x_1(t_k), x_2(t_k), x_3(t_k))$ are being determined. In (3.3): $\gamma_1(t_k), \gamma_2(t_k), \gamma_3(t_k)$ are three normally distributed independent random numbers for the k-th step, with means, equal to zero and variances, equal to 1 (The generation of above-mentioned numbers occurs separately); nq is a quantification number such that $1/nq = \sqrt{t_k - t_{k-1}}$ and when $nq \to \infty$, then the discrete process approaches the continuous Wiener process. In the implementation, the random process is simulated at each step of the walk and continues until it crosses the boundary.

It is known that there exist two principles for generating random numbers, physical and algorithmic:

1. The physical principle of generation gives truly random numbers, but its realization is expensive, especially in the multidimensional case, and therefore its application is not practical.

2. In spite of a great number of algorithmic methods, generating random numbers, they also have disadvantages which are contained in the generating principle itself, resulting a sequence of not truly, but pseudo-random numbers.

In this paper, when solving the Dirichlet boundary problems for Laplace's equation, we are using the pseudo-random numbers and their generation are performed in MATLAB environment.

4. An auxiliary classic problem

It should be noted that in 3D case there are no exact test solutions for generalized problems of type A. Therefore, for verification of the scheme needed for the numerical solution of Problem A, the reliability of obtained results can be demonstrated in the following way.

If we take $g_i(y) = 1/|y - x^0|$ in boundary conditions (2.5), where $y \in S_i(i = \overline{1,4}), x^0 = (x_1^0, x_2^0, x_3^0) \overline{\in} \overline{D}$, and $|y - x^0|$ denotes the distance between the points y and x^0, then it is evident that the curves $l_k(k = \overline{1,6})$ represent removable discontinuity curves for the boundary function $g(y)$. In the mentioned case instead of generalized problem A we obtain the next Dirichlet classical harmonic problem.

Problem B. Find a Function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D})$ under the following conditions:

$$\Delta u(x) = 0, \quad x \in D,$$
$$u(y) = 1/|y - x^0|, \quad y \in S, \ x^0 \overline{\in} \overline{D}.$$

We solve this problem by using the MPS with algorithm constructed for Problem A. It is known that Problem B is well posed, i.e., its solution exists, is unique and depends continuously on the data. An exact solution of Problem B has the form

$$u(x^0, x) = \frac{1}{|x - x^0|}, \ x \ \in \ \overline{D}, \ x^0 \overline{\in} \overline{D}. \tag{4.1}$$

Note that the process of solving the Dirichlet classical harmonic problems numerically by the MPS is quite interesting and important (see e.g., [15,16]). In the present paper, Problem B plays an auxiliary role and is used for checking the reliability of the scheme, and the corresponding program is needed for a numerical solution of Problem A. First, we solve Problem B and then compare the obtained results with the exact solution and solve Problem A under the boundary conditions (2.5).

In this paper the MPS is applied to one example. In the tables, N denotes a number of trajectories in the simulated Wiener process for the given points $x^i = (x_1^i, x_2^i, x_3^i) \in D$, and nq is a number of quantification. The tables below present for problem of type B the numerical absolute errors Δ^i at the points $x^i \in D$ of $u_N(x)$, in the MPS approximation, for various values of nq and N, and the numbers are given in scientific format. In particular,

$$\Delta^i = max|u_N(x^i) - u(x^0, x^i)|, \quad (i = 1, 2, \cdots, 5),$$

where $u_N(x^i)$ is the approximate solution of Problem B at the point x^i, which is defined by using formula (3.2), and the exact solution $u(x^0, x^i)$ of the test problem is given by (4.1). In the tables, for problems of type A, the probabilistic solution $u_N(x)$ is calculated at the points x^i, defined by (3.2).

Remark 2 : The Problems of type A and B for ellipsoidal, spherical, cylindrical, conic, prismatic, regular and irregular pyramidal, axis-symmetric finite domains with a cylindrical hole, external Dirichlet generalized problem for a sphere are considered in [12,14,16,17-21].

5. Numerical example

In order to determine the intersection points $y^j = (y_1^j, y_2^j, y_3^j)$ $(j = \overline{1, N})$ of the simulated process path and the surface S of P_3, first of all, for each current point $x(t_k)$ defined from (3.3) we check whether it belongs to P_3 or not.

Knowing the parameter h and coordinates of the vertices M, A_1, A_2, A_3, of P_3, we can: 1) write down equations of P_3 edges; 2) define angles of inclination $\alpha_1, \alpha_2, \alpha_3$ of the lateral faces with respect to the base of P_3; 3) write down equations of lateral faces.

Example 1. This example concerns such stationary problems which are used to determine the potential of an electric field or the temperature of a thermal field, and so on, in the domain D. In the role of D is taken interior of the irregular 3-sided pyramid $P_3(h)$, where h is its height.

As it was already noted above, we consider the case, when h is the lateral edge of P_3 and the angle between lateral faces containing h is acute. Besides, we assume that the base of P_3 lies in the first quarter of the plane Ox_1x_2, h lies on Ox_3, and $A_1 = (a, b, 0), A_2 = (0, c, 0), A_3 \equiv O = (0, 0, 0), M = (0, 0, h)$.

It is easy to see, that the equations of lines A_1A_2, A_1A_3, A_2A_3 are

$$((b-c)/a)x_1 - x_2 + c = 0,\ (b/a)x_1 - x_2 = 0,\ x_1 = 0 \ and \ x_3 = 0, \tag{5.1}$$

respectively and the equations of lines: A_1M, A_2M, A_3M are the following

$$h(x_1+x_2)+(a+b)x_3-h(a+b) = 0, (h/c)x_2+x_3-h = 0, x_1 = 0 \ and \ x_2 = 0, \tag{5.2}$$

accordingly.

First of all we must define angles $\alpha_1, \alpha_2, \alpha_3$ of inclination of the lateral faces of P_3: $S_1 = MA_1A_2; S_2 = MA_2A_3; S_3 = MA_3A_1$, with respect to the base of P_3.

It is clear that $\alpha_1 = arctan(h/\Delta_1)$, where Δ_1 is a distance between the point A_3 and the line A_1A_2. From the equation of line A_1A_2 (see (5.1)) $\Delta_1 = |c|/\sqrt{kk^2+1}$, where $kk = (b-c)/a$. Analogously, angles for the faces S_2 and S_3 are $\alpha_2 = \alpha_3 = \pi/2$.

It is not difficult to show that in the coordinate system $Ox_1x_2x_3$, the equations of faces S_1, S_2, S_3, S_4 are the following

$$\begin{gathered} S_1 : h(c-b)x_1 + ahx_2 + acx_3 - ahc = 0;\quad S_2 : x_1 = 0; \\ S_3 : bx_1 - ax_2 = 0;\quad S_4 : x_3 = 0. \end{gathered} \tag{5.3}$$

We have now the necessary information about the pyramid P_3 in order to establish whether each current point $x(t_k)$, defined from (3.3) belongs to P_3 or not. For this, we operate in the following way. For each step of simulated Wiener process we calculate angles $\beta_m(m=\overline{1,3})$ of inclinations of the planes passing through the points $x(t_k), A_m, A_{m+1}$ ($A_4 \equiv A_1$, with respect to the base of P_3. It is easy to see that

$$\beta_1 = \arctan(x_3(t_k)/dd), \beta_2 = arctan(x_3(t_k)/x_1(t_k)), \beta_3 = arctan(x_3(t_k)/dd^*),$$

where dd and dd^* are the distances between the point $(x_1(t_k), x_2(t_k))$ and the lines A_1A_2 and A_1A_3, respectively. On the basis of equations of lines A_1A_2 and A_1A_3 (see (5.1)) we have

$$dd = |kkx_1(t_k) - x_2(t_k) + c|/\sqrt{(kk)^2+1}, \;\; dd^* = |k1x_1(t_k) - x_2(t_k)|/\sqrt{(k1)^2+1},$$

where $kk = (b-c)/a$, $k1 = b/a$.

After calculating the angles β_m $(m = 1, 2, 3)$, we can compare them with angle $\alpha_m(m = 1, 2, 3)$. In particular:

(1*) if $\beta_m < \alpha_m$ and $0 < x_3(t_k) < h$ and $x_1(t_k) > 0$ and $x_2(t_k) > 0$ for $m = 1, 2, 3$ then the process is continued until it crosses the surface S;

(2*) if $\beta_1 = \alpha_1$ and $0 < x_3(t_k) < h$ and $x_1(t_k) > 0$ and $x_2(t_k) > 0$, then $x(t_k) \in \overline{S_1}$ or $y^j = (y_1^j, y_2^j, y_3^j) = x(t_k)$;

(3*) if $\beta_1 > \alpha_1$ and $0 < x_3(t_k) < h$, this means that the trajectory of the modulated Wiener process intersects the lateral face $S_1 \equiv A_1A_2M$ of P_3 or $x(t_{k-1}) \in D$ for the moment $t = t_{k-1}$ and $x(t_k) \;\overline{\in}\; P_3$ for the moment $t = t_k$. In this case, for approximate determination of the point y^j, a parametric equation of a line L passing through the points $x(t_{k-1})$ and $x(t_k)$ is obtained initially; it has the following form:

$$\begin{cases} x_1 = x_1(t_{k-1}) + (x_1(t_k) - x_1(t_{k-1}))\theta, \\ x_2 = x_2(t_{k-1}) + (x_2(t_k) - x_2(t_{k-1}))\theta, \\ x_3 = x_3(t_{k-1}) + (x_3(t_k) - x_3(t_{k-1}))\theta, \end{cases} \tag{5.4}$$

where (x_1, x_2, x_3) is a current point of L and θ is a parameter $(-\infty < \theta < \infty)$.

If we substitute the expressions of x_1, x_2, x_3, defined from (5.4), into (5.3) then we obtain the equations of faces S_1, S_2, S_3, S_4 with respect to θ, which have the following form

$$\begin{aligned} &S_1 : \theta = (ahc - h(c-b)x_1(t_{k-1}) - ahx_2(t_{k-1}) - acx_3(t_{k-1})) \\ &\qquad\quad /(h(c-b)C_1 + ahC_2 + acC_3), \\ &S_2 : x_1 = 0, \theta = -x_1(t_{k-1})/C_1, \\ &S_3 : \theta = (ax_2(t_{k-1}) - bx_1(t_{k-1}))/(bC_1 - aC_2), \\ &S_4 : x_3 = 0, \theta = -x_3(t_{k-1})/C_3, \end{aligned} \tag{5.5}$$

where $C_1 = x_1(t_k) - x_1(t_{k-1}), C_2 = x_2(t_k) - x_2(t_{k-1}), C_3 = x_3(t_k) - x_3(t_{k-1})$.

It is evident that on the basis of (5.5), if the intersection point $y^j \in S_i (i = \overline{1,4})$ then $y^j = (x_1(\theta), x_2(\theta), x_3(\theta))$, where θ is defined by (5.5), according to S_i.

Remark 3 : It is evident that probability of passing the path of simulated Wiener process through the discontinuous line, equals to zero, but if for any j the intersection point y^j lies on the discontinuous line (such case is taken into account in the calculation algorithm), then $g(y^j) = 0$ is taken in the role of j-th term of the series in formula (3.2).

In addition to the above, using equations (5.1) and (5.2) we can determine whether the intersection point y^j is on the edges of the pyramid or not.

Problems A and B are solved when $h = 2, a = 2, b = 2, c = 3, x^0 = (0.75, 1.5, -4)$. Since $a = b$, the angle between lateral faces containing h is to equal $\pi/4$. In Problem A the boundary function $g(y) \equiv g(y_1, y_2, y_3)$ has the following form

$$g(y) = \begin{cases} 1.5, & y \in S_1, \\ 2, & y \in S_2, \\ 1, & y \in S_3, \\ 3, & y \in S_4, \\ 0, & y \in l_k \ (k = \overline{1,6}). \end{cases} \tag{5.6}$$

In (5.6): $S_i(i = \overline{1,3})$ and S_4 are the lateral faces and the base of P_3 without discontinuity curves (edges), respectively; $l_k(k = \overline{1,6})$ are the edges of P_3. In the physical sense, l_k are non-conductors (or dielectrics).

In Example 1, considered by us for determination of the intersection points $y^j = (y_1^j, y_2^j, y_3^j)$ $(j = \overline{1,N})$ of the trajectory of a discrete Wiener process and the surface S, we have used the scheme, described above. As it was mentioned in Section 3, for the verification of calculating program for Problem A, firstly we solve the auxiliary Problem B.

Table 5.1B. Results for Problem B (in Example 1)

x^i	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 1)$	$(0.1, 0.2, 0.2)$	$(0.1, 0.2, 1)$	$(0.1, 0.2, 1.5)$
N	$\Delta^1, nq = 200$	$\Delta^2, nq = 200$	$\Delta^3, nq = 200$	$\Delta^4, nq = 200$	$\Delta^5, nq = 200$
$5E+3$	$0.46E-3$	$0.32E-3$	$0.10E-3$	$0.12E-3$	$0.47E-3$
$1E+4$	$0.19E-4$	$0.12E-3$	$0.62E-4$	$0.59E-5$	$0.18E-3$
$5E+4$	$0.26E-3$	$0.84E-4$	$0.50E-4$	$0.96E-4$	$0.18E-3$
$1E+5$	$0.18E-3$	$0.13E-4$	$0.20E-3$	$0.57E-4$	$0.12E-3$
$5E+5$	$0.19E-3$	$0.75E-4$	$0.19E-3$	$0.12E-3$	$0.53E-4$
$1E+6$	$0.20E-3$	$0.66E-4$	$0.95E-4$	$0.84E-4$	$0.11E-3$

In Table 5.1B the numerical absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at points $x^i \in D$ $(i = \overline{1,5})$ are presented for $nq = 200$ and various values of N. On the basis of obtained results, we can conclude that the calculating program for Problem A is correct.

Regardless of this, on the basis of analysis of obtained results, we come to the following: the accuracy is low at points x^1, x^3, x^4, x^5 and it does not improves when $N \to \infty$ (except the point x^2).

According to our opinion, the reason of indicated circumstances consists in: 1. the complexity of problem domain; 2. location of points $x^i (i = \overline{1,5})$ in the neighborhood of the domain surface.

The above mentioned inconvenience can be bypassed if for each point x^i we select the optimal, in the sense of accuracy, quantification number nq. The offered approach is correct theoretically (see Section 3). Thus, for each point x^i we solve the test problem B, e.g. for $nq = 50, 100, 200$ and so on. We stop this process if for some values of nq the accuracy of the corresponding probabilistic solution is enough for many practical problems, and the accuracy increases as $N \to \infty$.

For illustration, using the above described approach, for points $x^i (i = \overline{1,5})$ are selected two variants of numbers nq: I)(50,200,50,50,50); II)(400,200,100,100,100). In the Table 5.2B and Table 5.3B the numerical absolute errors Δ^i of the approximate solution $u_N(x)$ of the test Problem B at the point x^i for variants I) and II) are presented, respectively.

Table 5.2B. Results for Problem B (in Example 1)

x^i	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 1)$	$(0.1, 0.2, 0.2)$	$(0.1, 0.2, 1)$	$(0.1, 0.2, 1.5)$
N	$\Delta^1, nq = 50$	$\Delta^2, nq = 200$	$\Delta^3, nq = 50$	$\Delta^4, nq = 50$	$\Delta^5, nq = 50$
$5E+3$	$0.32E-3$	$0.47E-3$	$0.25E-4$	$0.48E-3$	$0.23E-3$
$1E+4$	$0.38E-4$	$0.57E-3$	$0.23E-3$	$0.29E-3$	$0.17E-3$
$5E+4$	$0.94E-4$	$0.17E-3$	$0.24E-5$	$0.23E-3$	$0.80E-4$
$1E+5$	$0.13E-4$	$0.67E-4$	$0.85E-4$	$0.96E-4$	$0.30E-4$
$5E+5$	$0.92E-4$	$0.69E-5$	$0.31E-4$	$0.27E-6$	$0.15E-4$
$1E+6$	$0.72E-4$	$0.19E-4$	$0.70E-4$	$0.10E-4$	$0.32E-4$

In the Table 5.2A the values of the approximate solution $u_N(x)$ to Problem A for the same points $x^i (i = \overline{1,5})$ and corresponding numbers nq are given (see Table 5.2B). The results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

Table 5.2A. Results for Problem A (in Example 1)

x^i	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 1)$	$(0.1, 0.2, 0.2)$	$(0.1, 0.2, 1)$	$(0.1, 0.2, 1.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.94840	1.68540	1.83020	1.68800	1.64160
$1E+4$	1.94460	1.68780	1.83715	1.68125	1.64940
$5E+4$	1.94041	1.68388	1.83458	1.68475	1.64744
$1E+5$	1.93784	1.68438	1.83559	1.68163	1.64615
$5E+5$	1.93989	1.68488	1.83309	1.68343	1.64916
$1E+6$	1.93987	1.68545	1.83307	1.68371	1.64941
$2E+6$	1.93994	1.68551	1.83423	1.68383	1.64831

In the Table 5.3B the numerical absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at the points x^i for variant II) are presented.

Table 5.3B. Results for Problem B (in Example 1)

x^i	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 1)$	$(0.1, 0.2, 0.2)$	$(0.1, 0.2, 1)$	$(0.1, 0.2, 1.5)$
N	$\Delta^1, nq = 400$	$\Delta^2, nq = 200$	$\Delta^3, nq = 100$	$\Delta^4, nq = 100$	$\Delta^5, nq = 100$
$5E+3$	$0.43E-3$	$0.62E-3$	$0.36E-3$	$0.30E-4$	$0.17E-3$
$1E+4$	$0.43E-4$	$0.11E-3$	$0.10E-3$	$0.21E-3$	$0.19E-3$
$5E+4$	$0.45E-4$	$0.21E-4$	$0.75E-4$	$0.11E-3$	$0.12E-3$
$1E+5$	$0.10E-3$	$0.27E-4$	$0.61E-4$	$0.30E-4$	$0.58E-4$
$5E+5$	$0.46E-4$	$0.16E-4$	$0.54E-4$	$0.23E-4$	$0.58E-4$
$1E+6$	$0.82E-5$	$0.64E-4$	$0.29E-4$	$0.21E-4$	$0.49E-4$

The values of approximate solution $u_N(x)$ to the Problem A for the same points $x^i (i = \overline{1,5})$ are given in Table 5.3A. The obtained results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

Table 5.3A. Results for Problem A (in Example 1)

x^i	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 1)$	$(0.1, 0.2, 0.2)$	$(0.1, 0.2, 1)$	$(0.1, 0.2, 1.5)$
N	$u_N(x^1)$,	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.93090	1.69030	1.84130	1.68660	1.65570
$1E+4$	1.95160	1.67800	1.83740	1.69745	1.65285
$5E+4$	1.94059	1.68813	1.83742	1.68840	1.65859
$1E+5$	1.94606	1.68632	1.83663	1.69337	1.65570
$5E+5$	1.94435	1.68563	1.83743	1.69100	1.65607
$1E+6$	1.94430	1.68630	1.83736	1.69062	1.65617

In this work we solved the problem of type A when boundary functions $g_i(y)(i = \overline{1,4})$ are constants. This was motivated by our interest to find out how well the obtained results agree with real physical picture. It is evident that solving Problem A under condition (2.5) is as easy as Problem B.

The analysis of the results of numerical experiments show that the results obtained by the proposed algorithm are reliable and it is effective for numerical solution of problems of type B and A. In particular, the algorithm is sufficiently simple for numerical implementation.

Besides, for the probabilistic solution $u_N(x)$ of Problems B and A, in the neighborhood of the domain surface, the method for selection optimal numbers nq is given.

6. Concluding remarks

1. This paper demonstrates that the suggested algorithm is ideally suited for numerical solution of problems B and A in such difficult domains as irregular pyramids.

2. According to this algorithm, there is no need to approximate the boundary function.

3. The computational outlays of this algorithm is low and the accuracy is sufficient for practical purposes.

4. The next steps of our research are related to:

* The numerical solution of Dirichlet classical and generalized harmonic problems for the infinite space R^3 with a finite number of spherical cavities.

* The MPS for the same type problem in finite domains which are bounded by several closed surfaces.

* The MPS for the same type problem in infinite 2D domains with a finite number of circular holes.

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