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BASIC OPERATIONS ON EXPONENTIAL MR-GROUPS

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Abstract. IIt is proved that the tensor completion is permutable with the operations of direct product and direct limit of exponential groups and, but in generally, is not permutable with the Cartesian product and the inverse limit of exponential groups.

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Introduction

The notion of an exponential R-group, where R is an arbitrary associative ring with unity, was introduced by R. Lyndon [1], Myasnikov and Remeslennikov [2] refined the notion of an R-group by introducing an additional axiom. The new concept of an R-group is a direct generalization of the concept of an R-module to the case of noncommutative groups. Amaglobeli and Remeslennikov [3] called R-groups with that additional axiom MR-groups. It turned out that all the previously studied Lyndon R-groups are in fact MR-groups (including Lyndon's free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$). In the present paper, we examine only those R-groups that are MR-groups. To simplify the notation, throughout what follows (unless specified otherwise), by an R-group we mean an MR-group, and denote the class of all R-groups by $\mathcal{M}_R$. The role of tensor extension of a scalar ring for modules is well known. An exact analog of this construction for an arbitrary R-group-tensor completion-was defined in [2]. A particular technique of constructing the tensor completion of a given R-group was offered in [4]. A systematic study of R-groups was started in [3-9]. Note that the results of these 3 papers have proven to be very useful in solving Tarski's well-known problems. In this paper [2] it is illustrated that the notion of a tensor completion plays the determining role in studying exponential groups. In this paper we investigate the question concerning the permutability a tensor completion functor with the basic groups operations.

1. Preliminary information and statements of problems

We recall basic definitions and facts following [1, 2]. Let $L_{gr} = \{\cdot,^{-1}, e\}$ be a group language (signature), where $\cdot$ is a binary multiplication operation,$^{-1}$ is a unary inversion operation, and e is a constant symbol for the identity of a group. In what follows, unless otherwise stated, R denotes an arbitrary fixed associative (possibly noncommutative!) ring with unity 1. We enrich the group language L_{gr} to a language $L^R_{gr} = L_{gr} \cup \{f_\alpha(g)|\alpha \in R\}$, where $f_\alpha(g)$ is a unary algebraic operation corresponding to raising to a power α.

Definition 1.1. [1]. A set G is called a Lyndon R-group if the operations $\cdot,^{-1}, e$, $\{f_\alpha(g)|\alpha \in R\}$ are defined on it, and the following axioms (below the expression $f_\alpha(g)$ is written as g^α for brevity) hold:

(1) group axioms;

(2) for all $g, h \in G$ and $\alpha, \beta \in R$

$$g^1 = g,\ g^0 = e,\ e^\alpha = e; \tag{1.1}$$

$$g^{\alpha+\beta} = g^{\alpha} \cdot g^{\beta}, \; g^{\alpha\beta} = (g^{\alpha})^{\beta}; \tag{1.2}$$

$$(h^{-1}gh)^{\alpha} = h^{-1}g^{\alpha}h; \tag{1.3}$$

Denote by $\mathcal{L}_R$ the category of Lyndon R-groups. Such groups will be called groups with exponents in R or R-exponential groups.

The axioms above are identities in the language L_{gr}^R, so the class $\mathcal{L}_R$ is a variety of algebraic systems in L_{gr}^R, and general theorems of universal algebra imply that we may well speak about R-homomorphisms, free R-groups, varieties of R-groups, quasivarieites of R-groups, and so on.

There exist Abelian Lyndon R-groups which are not R-modules; see [10] where the structure of a free Abelian Lyndon R-group is studied in detail. In [2], added to the Lyndon axioms is an extra axiom scheme-namely, the following series of quasi-identities:

$$(MR\text{-axiom}) \quad \forall g, h\, [g,h] = e \Rightarrow (gh)^{\alpha} = g^{\alpha}h^{\alpha}, \alpha \in R, \tag{1.4}$$

where $[g,h] = g^{-1}h^{-1}gh$.

Definition 1.2. [2]. A group $G \in \mathcal{L}_R$ is called an *MR*-group if G satisfies the *MR*-axiom (1.4).

Denote by $\mathcal{M}_R$ the class of all R-groups with exponents in R satisfying the *MR*-axiom (1.4). By definition, $\mathcal{L}_R \supset \mathcal{M}_R$ and, moreover, every Abelian R-group in $\mathcal{M}_R$ is an R-module and vice versa. The majority of natural examples of exponential R-groups belong to the class $\mathcal{M}_R$. An arbitrary group is a $\mathbb{Z}$-group in the class $\mathcal{M}_{\mathbb{Z}}$; a divisible Abelian group from $\mathcal{L}_{\mathbb{Q}}$ belongs to $\mathcal{M}_{\mathbb{Q}}$; a group of exponent n is a $\mathbb{Z}/n\mathbb{Z}$ -exponential *MR*-group; a free exponential Lyndon R-group is an R-exponential *MR*-group; an arbitrary pro-p-group is an exponential $\mathbb{Z}_p$-group in the class $\mathcal{M}_{\mathbb{Z}_p}$where $\mathbb{Z}_p$ is the ring of p-adic integers; for other examples, see [2].

In this paper, we study groups in the class $\mathcal{M}_R$. In what follows, therefore, unless otherwise stated, by an R-group we always mean a group in $\mathcal{M}_R$.

The class $\mathcal{M}_R$ is a quasivariety in the signature L_{gr}^R. Let $\mathcal{N}$ be some variety of $\mathcal{L}_R$-groups. Consider the intersection $\mathcal{N} \cap \mathcal{M}_{\mathcal{R}} = \mathcal{N}_{\mathcal{R}}$. The class $\mathcal{N}_R$ is also a quasivariety in the language L_{gr}^R, so it contains free R-groups and has a theory of defining relations. Moreover, $\mathcal{N}_R$ is closed under taking R-subgroups and we can compute R-factor groups in it [11]. Despite the fact that classes $\mathcal{N}_R$ are quasivarieties, we may well consider them as varieties within the quasivariety $\mathcal{M}_R$, i.e., as relative Birkhoff classes. For this reason, we find it convenient to call them varieties of R-groups.

Properly speaking, for any group $G \in \mathcal{L}_R$, concepts such as an R-subgroup, R-generatedness, a normal R-subgroup, etc., are introduced in a common way (see [2]). In particular, a homomorphism of R-groups $\varphi : G \to G^*$ is called an R-homomorphism if $\varphi(g^{\alpha}) = \varphi(g)^{\alpha}$ for any $g \in G$and any

$$\alpha \in R.$$

Definition 1.3. [2]. Let $G \in \mathcal{L}_R$. For $g, h \in G$ and $\alpha \in R$, the element

$$(g,h)_{\alpha} = h^{-\alpha}g^{-\alpha}(gh)^{\alpha}$$

is called an α-commutator of the elements g and h. It is straightforward to verify that for $G \in \mathcal{L}_R$, the following hold:

$$(gh)^{\alpha} = g^{\alpha}h^{\alpha}(g,h)_{\alpha}, \tag{1.5}$$

$$(g,h)_{-1} = [h^{-1}, g^{-1}], \tag{1.6}$$

$$f^{-1}(g,h)_{\alpha}f = \left(f^{-1}gf, f^{-1}hf\right)_{\alpha}, \tag{1.7}$$

$$G \in \mathcal{M}_R \Leftrightarrow ([g,h] = e \Rightarrow (g,h)_\alpha = e)\,. \tag{1.8}$$

The last equivalence leads to a definition of an $\mathcal{M}_R$-ideal.

Definition 1.4. [2]. Let $G \in \mathcal{L}_R$. A normal R-subgroup $H \trianglelefteq G$ is called an $\mathcal{M}_R$-ideal if, for any $g, h \in G$, the fact that $[g,h] \in H$ implies $(g,h)_\alpha \in H$ for any $\alpha \in R$.

In what follows, $\mathcal{M}_R$-ideals H in G are often called R-ideals and are denoted as $H \trianglelefteq_R G$. In [2], it was shown that if $\varphi : G \to G^*$ is an R-homomorphism of groups from $\mathcal{M}_R$, then $\ker\varphi$ is an R-ideal in G, and if H is an R-ideal in $G \in \mathcal{M}_R$, then $G/H \in \mathcal{M}_R$.

Proposition 1.1. *Let $G \in \mathcal{L}_R$.*

(1) *Intersection of any nonempty family of R-ideals in G is an R-ideal.*

(2) *For any subset $Y \subseteq G$, there exists at least (w.r.t. inclusion) R-ideal in G containing Y.*

Proof. The proof is standard.

Definition 1.5. Let $G \in \mathcal{L}_R$ and $Y \subseteq G$. Denote by $id_R(Y)$ the least (w.r.t. inclusion) R-ideal in G containing Y. The structure of R-ideals is clarified by the following:

Proposition 1.2. *Let $G \in \mathcal{L}_R$ and $Y \subseteq G$. Then $id_R(Y)$ is a union of the following ascending chain of R-subgroups in G:*

$$H_0 \le H_1 \le H_2 \le \cdots \le H_n \le\,,$$

where H_0 is a normal R-subgroup generated by Y, and

$$H_{i+1} = \langle H_i, (g_1,g_2)_\alpha |\, [g_1,g_2] \in H_i, \alpha \in R\rangle_R.$$

Note that all subgroups H_i, are normal subgroups in G.

Proof. By the definition of an R-ideal, all H_i, are contained in $id_R(Y)$. Equation (1.7) implies that all H_i, are normal R-subgroups in G, and so therefore their union $\bigcup_i H_i$ is a normal subgroup. Also by construction it is straightforward to verify that $\bigcup_i H_i$, is an R-ideal. Consequently, $id_R(Y) = \bigcup_i H_i$.

Proposition 1.3. *Let $G \in \mathcal{L}_R$ and $Y \subseteq G$. For any R-endomorphism if $\phi : G \to G$,*

$$\phi\,(id_R\,(Y)) \le id_R\,(\phi(Y))\,.$$

In particular, if ϕ is an R-automorphism, then

$$\phi\,(id_R\,(Y)) = id_R\,(\phi(Y))\,.$$

Proof. In the notation of Proposition 1.2, it suffices to show that for any$i \in \mathbb{N}$,

$$\phi\,(H_i\,(Y)) \le H_i\,(\phi\,(Y))\,,$$

where $H_i\,(Y)$ $(H_i\,(\phi\,(Y))$ resp.) is the subgroup H_i constructed in Proposition 1.2 for the set Y $(\phi\,(Y)$ resp.). Clearly, $\phi\,(H_0\,(Y)) \le H_0\,(\phi\,(Y))$. By induction, we can assume that $\phi\,(H_i\,(Y)) \le H_i\,(\phi\,(Y))$. The definition of $H_{i+1}\,(Y)$ implies that

$$\phi\,(H_{i+1}\,(Y)) \le \left\langle \phi\,(H_i\,(Y))\,, \phi\left((g_1,g_2)_\alpha\right) |\, [\mathrm{g_1,g_2}] \in \mathrm{H_i\,(Y)\,, \alpha \in R} \right\rangle_R \tag{1.9}$$

Note that $[g_1,g_2] \in H_i(Y)$ entails $\phi([g_1,g_2]) = [\phi(g_1), \phi(g_2)] \in H_i(\phi(Y))$. Consequently,

$$(\phi(g_1), \phi(g_2))_\alpha \in H_{i+1}(\phi(Y)). \tag{1.10}$$

Then

$$\begin{aligned}\phi\left((g_1,g_2)_\alpha\right) &= \phi({g_2}^{-\alpha}{g_1}^{-\alpha}(g_1g_2)^\alpha) \\ &= \phi(g_2)^{-\alpha}\phi(g_1)^{-\alpha}(\phi(g_1)\phi(g_2))^\alpha \\ &= (\phi(g_1), \phi(g_2))_\alpha.\end{aligned}$$

Now (1.10) yields $(\phi(g_1), \phi(g_2))_\alpha \in H_{i+1}(\phi(Y))$, and by (1.9), $\phi(H_{i+1}(Y)) \leq H_{i+1}(\phi(Y))$. It remains to use induction.

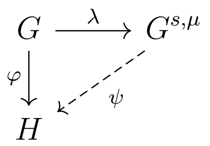
Corollary 1.1. Let $G \in \mathcal{L}_R$ and $Y \subseteq G$. If Y is invariant under all R-endomorphisms of G, then the ideal $id_R(Y)$ is invariant under all R-endomorphisms of G.

The tensor completion operation plays a decisive role in studying exponential R-groups. It naturally generalizes the notion of extension of a ring of scalars for modules to the non-commutative case. Tensor completion is used in defining free constructions in the class $\mathcal{M}_R$, including the concept of a free R-group.

Definition 1.6. Let G be an R-group and let $\mu : R \to S$ be a ring homomorphism. Then an S-group $G^{s,\mu}$is called the tensor S-completion of the R-group G if $G^{s,\mu}$ satisfies the following universal property:

(1) there exists a homomorphism $\lambda : G \to G^{s,\mu}$, compatible with μ (i.e., $\lambda(g^\alpha) = (\lambda(g))^{\mu(\alpha)}$ for all $g \in G$and $a \in R$) such that $\lambda(G)$ S-generates $G^{s,\mu}$, i.e. $\langle\lambda(G)\rangle_S = G^{s,\mu}$;

(2) for any S-group H and any homomorphism $\varphi : G \to H$ compatible with μ, there exists an S-homomorphism $\psi : G^{s,\mu} \to H$ making this diagram



commutative.

For a fixed ring homomorphism $\mu : R \to S$, group homomorphisms such as $\lambda$and φ from the definition above, compatible with μ, will further be called simply R-homomorphisms.

Notice that if G is an Abelian R-group, then

$$G^{S,\mu} \cong G \underset{R}{\otimes} S$$

is the tensor product of an R-module G and a ring S. In [2], it was proved that for any R-group G and any homomorphism $\mu : R \to S$, the tensor completion $G^{S,\mu}$. exists and is unique up to R-homomorphism.

Below the ring homomorphism $\mu : R \to S$ will be fixed, and so instead of $G^{S,\mu}$, in proofs we will use just the entry G^S. In applications, μ is most often a ring embedding, but in that case, also, an R-homomorphism $\lambda : G \to G^s$is not always an embedding. A general sufficient condition under which λ is an embedding can be found in [2, Prop. 11]. Groups that are isomorphically embedded in their tensor completion over a ring R were dealt with in [5].

Let $G \in \mathcal{L}_R$, $1 \in I$. Denote by $\overline{\prod} G_i$ and $\prod G_i$ the Cartesian and the groups G respectively. Let $G \in \overline{\prod} G_i$, $g = (\ldots, g_i, \ldots), \alpha \in R$. Define an action of R on G by coordinates

$$g^\alpha = (\ldots, {g_i}^\alpha, \ldots).$$

It can be immediately verified that if all the groups G_i, satisfy an axiom from (1.1)-(1.4) then the groups $\overline{\prod} G_i$ and $\prod G_i$ satisfy the same axiom. Thus, we have proved

Proposition 1.4. *The classes $\mathcal{L}_R$ and $\mathcal{M}_R$ are closed with respect to direct and Cartesian products.*

If in the standard definitions of direct and inverse spectrums one considers only R-homomorphisms then it is not difficult to prove

Proposition 1.5. *The classes $\mathcal{L}_R$ and $\mathcal{M}_R$ are closed with respect to direct and inverse limit.*

It is proved in [10] that in Abelian group category the operations of direct product of groups, of direct and inverse limits have a universal property. The corresponding actions in exponential group category have analogous properties. Here we restrict ourselves only to the formulation of corresponding universal properties.

Proposition 1.6. *(The universal property of direct products). Let $\varphi_i : G_i \to H$ be an H-homomorphisms, $i \in I$. Then the diagrams*

$$\begin{array}{ccc} G_i & \xrightarrow{\rho_i} & \prod\limits_i G_i \\ {\scriptstyle \varphi_i}\downarrow & \swarrow{\scriptstyle \psi} & \\ H & & \end{array} \qquad (i \in I),$$

where ρ_i are inclusion maps $[\varphi_i(G_i), \varphi_j(G_j)] = e, i \neq j$, it is possible to replace the dotted line by the uniquely defined R-homomorphism ψ (not depending on i) so that all the diagrams convert into commutative ones.

Denote by $G_* = \varinjlim G_i$ the limit group of the direct spectrum $\mathbb{G} = \{G_i(i \in I); \pi_i^j\}$. Given R-homomorphisms $\sigma_i : G_i \to H$ for which the diagrams

$$\begin{array}{ccc} G_i & \xrightarrow{\pi_i^j} & G_j \\ {\scriptstyle \sigma_i}\downarrow & \swarrow{\scriptstyle \sigma_j} & \\ H & & \end{array} \qquad (i \leq j);$$

are commutative, there exists one and only one homomorphism $\sigma : G_* \to H$ such that all the diagrams

$$\begin{array}{ccc} G_i & \xrightarrow{\sigma_i} & H \\ {\scriptstyle \pi_i}\downarrow & \nearrow{\scriptstyle \sigma} & \\ G_* & & \end{array} \qquad (i \in I,\ \pi_i \text{ is a projection of } G_i \text{ into } G)$$

are commutative.

Denote by $G_* = \varprojlim G_i$ the limit group of the inverse spectrum $\mathbb{G} = \{G_i(i \in I), \pi_i^j\}$.

Proposition 1.7. *(the universal property of inverse limits). Let H be an R-group and* *let $\sigma_i : H \to G_i$ be R-homomorphisms for which the diagrams*

$$\begin{array}{ccc} H & & \\ {\scriptstyle\sigma_j}\downarrow & \searrow^{\sigma_i} & \\ G_j & \xrightarrow[\pi_i^j]{} & G_i \end{array} \qquad (i \le j)$$

are commutative, there exists one and only one homomorphism $\sigma : H \to G^*$ such that all the diagrams

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & G^* \\ & \searrow & \downarrow{\scriptstyle\pi_i} \\ & & G_i \end{array} \qquad (i \in I,\ \pi_i \text{ is a projection of } G^* \text{ into } G_i)$$

are commutative.

2. Construction of tensor completion

In this section we come up with a method for constructing tensor completion using the apparatus of combinatorial group theory [5,9].

Case 1. Let $\mu : R \to S$ be an epimorphism. Then $S = R/M$, where $M = \ker \mu$. Let G be an arbitrary MR-group,

$$G_0 = \{g \in G |\, g = f^\alpha \text{ for some } f \in G, \alpha \in R\},$$

and let $G_\mu = id(G_0)$ be an $\mathcal{M}_R$-ideal generated by G_0. Then the factor group $\overline{G} = G/G_\mu$ is an MS-group under the induced action of S on $\overline{G} = (gG_\mu)^\beta = g^\alpha G_\mu$, where α is an element such that $\mu(\alpha) = \beta$. Denote by $\lambda : G \to G/G_\mu$, the canonical homomorphism of Gonto G/G_μ.

Theorem 2.1. *Let $\mu : R \to S$ be a ring epimorphism. Then $G^S \cong G/G_\mu$ where MS-group G/G_μ is defined as above.*

Proof. Let $\varphi : G \to H$ be an R-homomorphism of G onto an MR-group H consistent with μ. Then $\ker\varphi \ge G_\mu$, and so there exists an S-homomorphism $\psi : G^S \to H$ making the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\lambda} & G^S \\ {\scriptstyle\varphi}\downarrow & \swarrow_{\exists\psi} & \\ H & & \end{array}$$

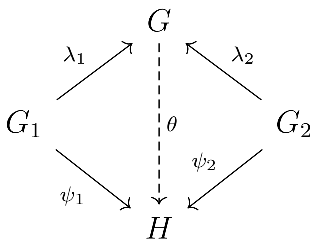
Example. Let $R = \mathbb{Z}$ be the ring of integers, $S = \mathbb{Z}_n$the ring of integers modulo n, $\mu : \mathbb{Z} \to \mathbb{Z}_n$ the natural homomorphism. Then let $G_\mu = G^n$is a subgroup of G generated by n-powers of elements in G and $G^S \cong G/G^n$ is a maximal factor group of G of period n.

Case 2. Let μ be an embedding. Suppose G is a partial MR-group and $\mu : R \to S$ is a ring embedding. We describe how to construct tensor S-completion for G. Recall the definition of a free product with amalgamated subgroup (see, e.g., [12]).

Let $h_i \leq G_i$ be groups, $i = 1, 2$,and let an epimorphism $\varphi : H_1 \to H_2$ be fixed. We call G the free product of groups G_1 and G_2 with subgroups H_1 and H_2 amalgamated with respect to φ, and we define $G = *\,(G_1, G_2, H_1, H_2, \varphi)$ if G satisfies the following universal property:

(a) there exist homomorphisms $\lambda_1 : G_1 \to G$ and , $\lambda_2 : G_2 \to G$ such that G is generated by $\lambda_1(G_1)$ and $\lambda_2(G_2)$;

(b) for any group H and for any homomorphisms $\psi_1 : G_1 \to H$ and $\psi_2 : G_2 \to H$ consistent with φ, there exists a homomorphism $\theta : G \to H$ such that the diagram



$(\theta \circ \lambda_1 = \psi_1,\ \theta \circ \lambda_2 = \psi_2)$

is commutative. If $G = \langle X_1 | \mathrm{R}_1 \rangle$ and $G = \langle X_2 | \mathrm{R}_2 \rangle$ are presentations of G_1 and G_2 by generators and defining relations, then

$$G = \langle X_1 \cup X_2 | \mathrm{R}_1 \cup \mathrm{R}_2 \cup \mathrm{T} \rangle\,,$$

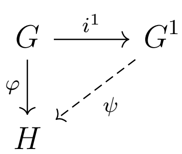
where $T = \{\varphi\,(h_1) = h_2 |\, \forall h_1 \in H_1\}$ is a presentation of G by generators and defining relations.

We embark on the construction of tensor completion G^S proceeding in steps.

Elementary step. Let M be a maximal Abelian subgroup of a partial *MR*-group G. Consequently, M is a partial R-module and, therefore, a partial S-module. Denote $M^S \equiv M \underset{S}{\otimes} S$ by M^S. Then M^S is an S-module, and we let $i_M : M \to M^S$ be a canonical map, which is a partial R-homomorphism. Put $G_0^1 = *(G, M^S, M, i_M(M), i_M)$.

For the image $\lambda_1(g)$ of an element $g \in G$ in the group G_0^1, raising to a power $\alpha \in S$ is defined by a formula $\lambda_1(g)^\alpha = \lambda_1\,(g^\alpha)$ for those $\alpha \in S$ for which g^α is defined in the group G. Similarly, we define a partial action of S on the image $\lambda_2(M^S)$ in G_0^1 of the group M^S. It is easy to see that the thus defined action is correct and the group G_0^1 is a partial group in the class $\mathcal{PK_S}$ (axioms (1.1) and (1.2) are satisfied). Let $N = id_{\mathcal{M}}\,(1)$, i.e, the least $\mathcal{M}$-ideal in G_0^1 making the factor group G_0^1/N a partial *MS*-group. Let $G^1 = G_0^1/N$ and $\eta : G_0^1 \to G_0^1/N$ be the natural homomorphism. Denote by $i^1 : G \to G^1$ partial R-homomorphism induced by the maps $i^1 : G \to G_0^1$ and $\eta : G_0^1 \to G_0^1/N$. We say that the group G^1 is obtained from G by an elementary $\mathcal{M}$-step using a subgroup M. Similarly, we can define the concept of an elementary $\mathcal{L}$-step in the category $\mathcal{PL_S}$.

Lemma 2.1. *(homomorphism extension). Let φ be a partial R-homomorphism of an MR-group G into an MS-group H. Then there exists a partial S-homomorphism $\psi : G^1 \to H$ such that the diagram*



is commutative.

Proof. By construction, the group G_0^1 is generated by the subgroups $i^1(G)$ and $\eta(M^S)$. For $i^1(g), g \in G$, put $\psi(i^1(G)) = \varphi(g)$. The restriction of to M induces a homomorphism

$M \to H$ and, therefore, an S-homomorphism $\varphi_M : M^S \to H$ consistent with $i_M : M \to M^S$. The universal property of free products with amalgamated subgroup yields a homomorphism $\psi_0 : G \longrightarrow H$ extending the maps φ and φ_M. Since H is an MS-group, the $\mathcal{M}$ -ideal n lies in the kernel of ψ_0 Then ψ_0 induces the desired partial S-homomorphism $\psi : G^1 \to H$.

Nonlimit step. The outcome of a first elementary step is that we managed to raise the images of elements of the maximal Abelian subgroup M to powers in the ring S. In the noncommutative case, it is natural to continue the construction by taking a second step, a third step, ..., a kth step, $k \in \mathbb{N}$, and obtain groups G^k and partial S-homomorphisms $i^k : G^{k-1} \to G^k$. The last homomorphisms make it possible to define a partial S-homomorphism $\pi_r^s : G^r \to G^s$ for any index pair (r, s), $r < s$. The system $\mathbb{G} = \{G | k \in N, \pi_r^s (r < s)\}$ is a direct spectrum.

Limit step. Let G^ω be the limit group of the direct spectrum $\mathbb{G}$ and let π_k^ω be the projection of a group G^k into G^ω. The homomorphism extension lemma is naturally proved also for the group G^ω.

(It suffices to recall the universal property of a direct limit). If the group G^ω is not an MS-group, then we continue taking the following steps: $G^{\omega+1} = (G^\omega)^1$, $G^{\omega+2} = (G^{\omega+1})^1, G^{\omega+3} = (G^{\omega+2})^1, \ldots$. The procedure can always be arranged so that there exists an ordinal v with which G^v is already MS-group. At every step conditions (1) and (2) in the definition of a group G^S were satisfied, so G^v is the tensor S-completion for G.

3. Commutation of the functor of tensor completion with group operations

In this section, we examine the questions of the commutation of the tensor completion with the operations of direct and Cartesian products and direct and inverse limits.

Theorem 3.1. *The functor of tensor completion is permutable with a direct product. In other words, if* $G = \prod_j G_j$ *then*

$$G^S = \prod_j {G_j}^S \tag{3.1}$$

Before proving the theorem, let us formulate and prove the following lemma.

Lemma 3.1. *Let for each* j *there exist an* R*-homomorphism* $\varphi_i : G_i \to H$*, where* H *is an* R*-group, and also*

$$[\varphi_i (G_i), \varphi_j (G_j)] = e \tag{3.2}$$

for all pairs i, j*, where* $i \neq j$*. Then there exists an* R*-homomorphism* $\psi : \prod_i G_i \to H$ *that continues* φ_i *for all* i*.*

Proof. Denote $H_0 = \langle \varphi_i (G_i), i \in I \rangle_R \leq H$. Let us perform the linear ordering of the set of indices I and prove that any element $h \in H_0$ is representable in the form

$$g_{i_1} g_{i_2} \ldots g_{i_s}, \text{ where } g_{i_k} {\in} \varphi_{i_k}(G_{i_k}),\ i_1 < i_2 < \cdots < i_j \tag{3.3}$$

By virtue of condition (2) it is obvious that elements of form (3.2) make a subgroup. Since $\forall \alpha \in R$ by axiom (1.4) the element

$$(g_{i_1} g_{i_2} \ldots g_{i_s})^\alpha = {g_{i_1}}^\alpha {g_{i_2}}^\alpha \ldots {g_{i_s}}^\alpha$$

elements of form (3.3) make an R -subgroup. Now the required R -homomorphism can be constructed as follows. Let an element $g \in \prod_i G_i$, be written in the form

$$g = (\ldots, g_{i_1} g_{i_2} \ldots g_{i_s}, \ldots)$$

where instead of the points there are units. Assume that

$$\psi(g) = g_{i_1} g_{i_2} \dots g_{i_s}.$$

That ψ is a homomorphism can be verified in a straightforward manner.

Let us return to the proof of the theorem. For this, between the groups G^S and $\prod_i G_i^S$ we construct a pair of counter R -homomorphisms. Let

$$\lambda_i : G_i \to G_i^S$$

be a canonical R -homomorphism given by the definition of tensor completion. By Lemma 3.1 there exists an R -homomorphism

$$\varphi : G \to \prod_i G_i^S.$$

Then it is obvious that $[\lambda_i (G_i), \lambda_j (G_j)] = e$, if $i \neq j$. By the definition of tensor completion there exists a S-homomorphism ψ_1 that makes the diagram

$$\begin{array}{ccc} G = \prod_i G_i & \xrightarrow{\lambda} & G^S \\ \downarrow & \swarrow{\scriptstyle \exists\psi_1} & \\ \prod_i G_i^S & & \end{array}$$

commutative.

Let us verify that

$$[\lambda_i (G_i), \lambda_j (G_j)] = e.$$

This is so because the generatrices of the first subgroup commute with the generatrices of the second subgroup. By virtue of the definition of tensor completion there exists a S-homomorphism β_i, that makes the diagram

$$\begin{array}{ccc} G = \prod_i G_i & \xrightarrow{\lambda} & \langle \lambda(G_i) \rangle_S \\ \downarrow & \swarrow{\scriptstyle \beta_i} & \\ G_i^S & & \end{array} \qquad (i \in I)$$

commutative. Now by virtue of Lemma 3.1 there exists a S-homomorphism $\psi_2 : \prod_i G_i^S \to G^S$.

That these are counter-homomorphisms can be verified in a straightforward manner.

Theorem 3.2. *The operation of tensor completion is permutable with direct limits.*

Proof. Let us construct in a standard manner the counter-mappings between the group G_*^S and the group

$$H = \varinjlim G_i^S.$$

Denote

$$\varphi_i : G_i^S \to H$$

Then there exists a S-homomorphism

$$\psi_1 : H \to G_*^S$$

that makes the diagram

$$\begin{array}{ccc} G_i^S & \xrightarrow{\pi_i^S} & G_*^S \\ \downarrow & \swarrow_{\exists \psi_1} & \\ H & & \end{array}$$

commutative. Here $\pi_i : G_i \to G_*$ is the projection, $\pi_i^S : G_i^S \to G_*^S$ is the corresponding homomorphism of tensor completions. Using the universal property of tensor completion, for every index i we construct the mapping $(\psi_2)_i$, that makes the diagram

$$\begin{array}{ccc} G_i^S & \xrightarrow{\pi_i^S} & G_*^S \\ \psi_i \downarrow & \swarrow_{(\psi_2)_i} & \\ H & & \end{array}$$

commutative. Since the subgroups $\pi_i(G_i)$ cover the group G_* and since the homomorphisms $(\psi_2)_i$and $(\psi_2)_j$are consistent on the common elements, the R-homomorphism $\psi_2 : G_* \to H$is well-defined. The S-homomorphism will be the sought counter homomorphism for ψ_1.

Remark 1. The permutability of tensor completion with direct limits allows one to reduce many problems on completion to the case of a finitely generated group. In fact, let $\{G_i, (i \in I), \pi_i^j\}$ be the direct limit of all finitely generated subgroups of the group G. Then $G = \varinjlim G_i$ and $G^S \cong \varinjlim {G_i}^S$.

Remark 2. Let us give an example showing that the Cartesian product operation is not permutable with the operation of tensor completion. Denote

$$\lambda : \overline{\Pi}_i G_i \to \overline{\Pi} G_i^S.$$

Then by virtue of the universal property of tensor completions we have the S-homomorphism

$$\lambda^S\colon \left(\overline{\Pi}_i G_i\right)^S \to \overline{\Pi} G_i^S$$

which in the general case is not an isomorphism. An analogous example already exists in the theory of abelian groups.

Let us take as a ring R the field of rational numbers $\mathbb{Q}$, while as G_n, $n \in \mathbb{N}$, we take a cyclic group of order n. Let $G_n = \langle a_n \rangle$, $n \in \mathbb{N}$. Then

$$G_n^{\mathbb{Q}} = G_n \otimes \mathbb{Q} = 0.$$

Therefore

$$(\overline{\Pi} G_n)^{\mathbb{Q}} = 0.$$

At the same time the group $\overline{\Pi}_n G_n$, contains elements of infinite order and therefore the group

$$(\overline{\Pi} G_n)^{\mathbb{Q}} = \overline{\Pi} G_n \otimes \mathbb{Q}$$

is nonzero.

Let G^* be a limit group of the inverse spectrum

$$G = \{G_i, (i \in I), \pi_i^j\}.$$

We construct the S-homomorphism

$$\sigma(G^*)^S \to \varprojlim G_i^S.$$

For this we denote by

$$\pi_i : G^* \longrightarrow G_i$$

the projection of the limit group onto the i -th component. Then

$$\pi_i^S : (G^*)^S \to G_i^S$$

is the corresponding homomorphism of tensor completions. Let

$$\mu_i : \varprojlim G_i^S \to G_i^S$$

be the natural projection. Then there exists a homomorphism

$$\sigma(G^*)^S \to \varprojlim G_i^S,$$

that makes the diagram

$$\begin{array}{ccc} (G^*)^S & \overset{\exists\sigma}{\dashrightarrow} & \varprojlim G_i^S \\ {\scriptstyle \pi_i^S}\downarrow & \swarrow {\scriptstyle \mu_i} & \\ G_i^S & & \end{array}$$

commutative.
We will illustrate by an example that in the general case this homomorphism is not an isomorphism. Let us consider G_k, $k \in \mathbb{N}$, $G_k = \langle a_k \rangle$, where a_k is an element of order p^k, p is a prime number. Then, as is known, $\varprojlim G_k \cong \mathbb{Z}_{p^\infty}$ is an additive group of integer p-adic numbers,

$$\mathbb{Z}_{p^\infty}^{\mathbb{Q}} = \mathbb{Z}_{p^\infty} \otimes \mathbb{Q}$$

is the vector space over $\mathbb{Q}$ of continual cardinality. Simultaneously,

$$\varprojlim {G_k}^{\mathbb{Q}} = \varprojlim (G_k \otimes \mathbb{Q}) = \varprojlim 0 = 0.$$

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