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OSCILATION CRITERIA FOR FIRST-ORDER DIFFERENTIAL EQUATIONS WITH DELAY ARGUMENT

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Abstract. In the work the following differential equation

$$x'(t) + p(t)\,x(\sigma(t)) = 0, \quad t \geq 0$$

is considered, where $p \in L_{loc}(R_+, R_+)$, $\sigma \in C(R_+; R)$, $\sigma(t) \leq t$, for $t \in R_+$ and $\lim_{t\to+\infty} \sigma(t) = +\infty$. New oscillation criteria of all solutions to this equation are established.

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1. Introduction

In the present paper, using the ideas contained in [1], we establish some new oscillation criteria for the equation

$$x'(t) + p(t)\,x(\sigma(t)) = 0, \tag{1.1}$$

to be oscillatory, where

$$p \in L_{loc}(R_+; R_+), \;\; \sigma \in C_{loc}(R_+; R), \;\; \sigma(t) \leq t \;\text{ for }\; t \in R_+ \tag{1.2}$$

and

$$\lim_{t\to+\infty} \sigma(t) = +\infty.$$

It is a trivial fact that a first order linear ordinary differential equation cannot have oscillatory solutions (see definition below). Specific criteria for the oscillation of linear delay differential equations were for the first time proposed by A.D. Mishkis (see [2]). The oscillatory properties all proper solutions of delay differential equations, see in monographs [3,4] and the references therein.

Theorem 1.1 ([5] *Theorem* 1). *If*

$$\liminf_{t\to+\infty} \int_{\sigma(t)}^{t} p(s)\,ds > \frac{1}{e}\,,$$

then all proper solutions of equation (1.1) *are oscillatory.*

Theorem 1.2 ([6] *Theorem* 3). *If*

$$\int_{\sigma(t)}^{t} p(s)\,ds \leq \frac{1}{e} \quad \textit{for large} \quad t, \tag{1.3}$$

then equation (1.1) *has an eventually nonoscillatory solution.*

In particular, Theorem 1.2 can be formulated in the following way

Theorem 1.2′. *If*

$$\limsup_{t\to+\infty}\int_{\sigma(t)}^{t}p(s)\,ds<\frac{1}{e}\,, \tag{1.4}$$

then equation (1.1) *has an eventually nonoscillatory solution.*

Remark 1.1. We show below (Theorem 2.3) that in some case, the condition

$$\limsup_{t\to+\infty}\int_{\sigma(t)}^{t}p(s)\,ds>\frac{1}{e}\,, \tag{1.5}$$

is sufficient for every proper solution of equation (1.1) to be oscillatory.

Below we will use the following notation:

$$t_0\in\mathbb{R}_+,\quad \delta(t)=\max\{\sigma(s):s\in[t_0,t]\},\quad \eta(t)=\max\{s:\sigma(s)\le t\}$$

for $t\in\mathbb{R}_+$, $\eta_1(t)=\eta(t)$, $\eta_i=\eta_1\circ\eta_{i-1}$, $(i=2,3,\dots$.

Definition 1.1. Let $t_0\in\mathbb{R}_+$. A continuous function $x:[t_0,+\infty)\to\mathbb{R}$ is said to be a proper solution of equation (1.1), if it is locally absolutely continuous on $[\eta_1(t_0),+\infty)$, satisfies (1.1) almost everywhere in $\eta_i(t),+\infty)$ and $\sup\{|x(s):t\le s<+\infty\}>0$ for $t\ge t_0$.

Definition 1.2. A proper solution of equation (1.1) is said to be oscillatory, if the set of its zeros is unbounded from above. Otherwise it is said to be nonoscillatory.

Definition 1.3. Equation (1.1) is said to be oscillatory, if any of its proper solutions is oscillatory.

Lemma 1.1. *Let*

$$h\in C^{(1)}(\mathbb{R}_+),\quad h'(t)>0\quad \textit{for}\quad t\in\mathbb{R}_+\quad \textit{and}\quad \lim_{t\to+\infty}h(t)=+\infty. \tag{1.6}$$

Let moreover

$$h(\sigma(t))\ge\sigma(h(t))\quad \textit{for}\quad t\in\mathbb{R}_+ \tag{1.7}$$

and equation (1.1) *has an eventually positive solution. Then the differential inequality*

$$y'(t)+p_h(t)\,y(\sigma(t))\le 0 \tag{1.8}$$

also has an eventually positive solution, where

$$p_h(t)=h'(t)\,p(h(t)). \tag{1.9}$$

Proof. Suppose that equation (1.1) has an eventually positive solution $x:[t_0,+\infty)\to(0,+\infty)$. Then by (1.6) there exists $t_1\ge t_0$ such that

$$x'\big(h(t)\big)+p\big(h(t)\big)\,x\big(\sigma\big(h(t)\big)\big)=0\quad\text{for}\quad t\ge t_1. \tag{1.10}$$

Put $y(t)=x(h(t))$. Since $y'(t)=h'(t)\,x'(h(t))$, from (1.10) we have

$$y'(t)+h'(t)\,p(h(t))\,x\big(\sigma(h(t))\big)=0\quad\text{for}\quad t\ge t_1.$$

Therefore, since $x(t)$ is a decreasing function, by (1.7)we have

$$y'(t)+h'(t)\,p\big(h(t)\big)\,y\big(\sigma(t)\big)\le 0\quad\text{for}\quad t\ge t_1.$$

Hence $y(t) = x(h(t))$ is an eventually positive solution of inequality (1.8), where $p_h(t)$ is defined by (1.9).

Corollary 1.1. Let

$$\sigma \in C^{(1)}(\mathbb{R}_+) \quad \text{and} \quad \sigma'(t) \leq 1 \quad \text{for} \quad t \geq t_0, \tag{1.11}$$

and the equation (1.1) has an eventually positive solution. Then for any $c > 0$, the inequality

$$y'(t) + p(t+c)\, y(\sigma(t)) \leq 0 \tag{1.12}$$

also has an eventually positive solution.

Proof. According to (1.1), it is clear that (1.7) is fulfilled, where $h(t) = t + c$ and $c \in \mathbb{R}_+$, which proves the validity of the corollary.

2. Differential inequalities

Consider the differential inequality

$$y'(t) \operatorname{sign} y(t) + p(t)\big|y(t)\big| \leq 0, \tag{2.1}$$

where the functions p and σ satisfy conditions (1.2). Using the ideas contained in [1], we establish some oscillation criteria for the differential inequality (2.1).

Remark 2.1. A proper solution of the differential inequality is defined as in the case of a differential equation.

Theorem 2.1. *Let for some* $n \in \mathbb{N}$

$$\limsup_{t\to+\infty} \int_{\delta(t)}^{t} p(s) \, \exp\left\{ \int_{\delta(\xi)}^{\delta(t)} p(s)\, \psi_n(s)\, ds \right\} d\xi > 1, \tag{2.2}$$

where

$$\psi_1(t) = 1, \quad \psi_k(t) = \exp\left\{ \int_{\sigma(t)}^{t} p(\xi)\, \psi_{k-1}(\xi)\, d\xi \right\} \quad k = 2, \dots, n. \tag{2.3}$$

Then the differential inequality (2.1) *is oscillatory.*

Proof. Suppose, the contrary, that inequality (2.1) has a nonoscillatory solution $y[t_0, +\infty) \to R$. It can be assumed without loss of generality that $y : [t_0, +\infty) \to (0, +\infty)$ is eventually a solution of inequality (2.1) and

$$y'(t) + p(t)\, y\big(\sigma(t)\big) \leq 0 \quad \text{for} \quad \eta_1(t_0) \leq t \leq s < +\infty. \tag{2.4}$$

From (2.4) we get

$$y(s) \geq \exp\left\{ \int_{t}^{s} p(\xi)\, \frac{y(\sigma(\xi))}{y(\xi)} \right\} y(t) \quad \text{for} \quad t \geq \eta(t_0)$$

and

$$y\big(\sigma(t)\big) \geq \exp\left\{ \int_{\sigma(t)}^{t} p(\xi)\, \frac{y(\sigma(\xi))}{y(\xi)} d\xi \right\} y(t) \quad \text{for} \quad t \geq \eta_1(t_0). \tag{2.5}$$

Hence by (2.5) we have

$$y\big(\sigma(t)\big) \geq \Psi_j(t)\, y(t) \quad \text{for} \quad t \geq \eta_i(t_0) \quad (i = 1, \dots, n), \tag{2.6}$$

where the functions $\Psi_i(t)$ are given by (2.3).

According to (2.1) and (2.6) there is $t_* > \eta(t_0)$ such that

$$\int_{\sigma(t_*)}^{t_*} p(\xi)\exp\left\{\int_{\delta(\xi)}^{\sigma(t_*)} p(s)\Psi_n(s)ds\right\}d\xi > 1. \tag{2.7}$$

On the other hand, from (2.4) and (2.5), because δ is a nondecreasing function, we have

$$y\big(\delta(t)\big) \geq \exp\left\{\int_{\delta(t)}^{\delta(t_n)} p(\xi)\,\Psi_n(\xi)d\xi\right\}y\big(\delta(t_*)\big) \quad \text{for} \quad \delta(t_*) \leq t \leq t_*.$$

Since $y(\sigma(t)) \geq y(\delta(t))$, from (2.3) by (2.8) we get

$$y'(t) + p(t)\exp\left\{\int_{\delta(t)}^{\delta(t_*)} p(\xi)\Psi_n(\xi)d\xi\right\}y\big(\delta(t_*)\big) \leq 0. \tag{2.8}$$

Integrating (2.8) from $\delta(t_*)$ to t_*, we obtain

$$y(t_*) + y\big(\delta(t_*)\big)\left(\int_{\delta(t_*)}^{t_*} p(t)\exp\left\{\int_{\delta(t)}^{\delta t_*} p(s)\Psi_n(s)ds\right\}dt - 1\right) \leq 0.$$

This however contradicts (2.7). The obtained condradiction proves that inequality (2.1) is oscillatory.

Corollary 2.1 Let

$$\limsup_{t\to+\infty}\int_{\delta(t)}^{t} p(\xi)\exp\left\{\int_{\delta(\xi)}^{\delta(t)} p(s)ds\right\}d\xi > 1.$$

Then the differential inequality (2.1) is oscillatory.

Theorem 2.2 *Let*

$$\liminf_{t\to+\infty}\int_{\sigma(t)}^{t} p(s)ds > \frac{1}{e}\,. \tag{2.9}$$

Then the differential inequality (2.1) *is oscillatory.*

Proof. To prove the theorem, if suffices to show that from (2.10) it follows (2.2). Note that condition (2.10) implies

$$\liminf_{t\to+\infty}\int_{\sigma(t)}^{t} p(\xi)d\xi > \frac{1}{e}\,. \tag{2.10}$$

By (2.10), there exist $t_0 \in R_+$ and $c > \frac{1}{e}$ such that

$$\int_{\sigma(t)}^{t} p(\xi)d\xi \geq c \quad \text{for} \quad t \geq t_0\,. \tag{2.11}$$

By (2.11), there exist $t_0 \in R_+$ and $c > \frac{1}{c}$ such that

$$\int_{\sigma(t)}^{t} p(\xi)d\xi \geq c \quad \text{for} \quad t \geq t_0\,. \tag{2.12}$$

Choose a natural number n_0 such that $(ec)^{n_0} > \frac{4}{c^2}$. Because $\delta(t) \geq \sigma(t)$, from (2.12) we obtain $\psi_i(t) \geq (ec)$ $(i = 1, 2, \ldots, n_0)$, for large t. Therefore

$$\int_{\delta(t)}^{t} p(s) \exp\left\{ \int_{\delta(s)}^{\delta(t)} p(\xi)\, \psi_{n_0}(\xi) d\xi \right\} ds$$

$$\geq (ec)^{n_0} \int_{\delta(t)}^{t} p(s) \left(\int_{\delta(s)}^{\delta(t)} p(\xi) d\xi \right) ds \quad \text{for} t \geq t_1. \tag{2.13}$$

On the other hand, by (2.12), for any $t \geq t_1$, there exists $t^* \in [\delta(t), t]$ such that

$$\int_{\delta(t)}^{t^*} p(s) ds = \frac{c}{2} \quad \text{and} \quad \int_{\delta(t^*)}^{\delta(t)} p(s) ds \geq \frac{c}{2} \quad \text{for} \quad t \geq t_1.$$

Since

$$\int_{\delta(t)}^{t} p(s) \exp\left\{ \int_{\delta(s)}^{\delta(t)} p(\xi)\, \psi_n(\xi) d\xi \right\} ds$$

$$\geq (ec)^{n_0} \int_{\delta(t)}^{t} p(s) ds \left(\int_{\delta(s)}^{\delta(t)} p(\xi) d\xi \right) ds \geq (ec)^{n_0} \frac{c^2}{4} > 1\,,$$

which means that (2.2) is fulfilled.

Theorem 2.3 *Let*

$$\sigma \in C^{(1)}(R_+), \quad \sigma'(t) \leq 1 \quad \textit{for} \quad t \in \mathbb{R}_+, \tag{2.14}$$

and for a large $c > 0$

$$\liminf_{t \to +\infty} \left(\int_{\sigma(t)+c}^{t+c} p(s) ds - \int_{\sigma(t)}^{t} p(s) ds \right) \geq 0. \tag{2.15}$$

Then the condition

$$\limsup_{t \to +\infty} \int_{\sigma(t)}^{t} p(s) ds > \frac{1}{e} \tag{2.16}$$

is sufficient for equation (1.1) to be oscillatory.

Proof. According to (1.16), there exist $\{t_k\}_{k=1}^{+\infty}$ and $\varepsilon > 0$, such that

$$\int_{\sigma(t_i)}^{t_i} p(s) ds \geq \frac{1+\varepsilon}{e} \quad i = 1, 2, \ldots\,. \tag{2.17}$$

Suppose, to the contrary, that there exists an eventually positive solution $x : [t_0, +\infty) \to (0, +\infty)$ of equation (1.1). By Corollary 1.1, the differential inequality

$$y'(t) + p(t + t_i)\, y(\sigma(t)) \leq 0 \tag{2.18}$$

has an eventually positive solution.

Hand by (2.17)

$$\int_{\sigma(t)}^{t} p(s + t_i) ds = \int_{\sigma(t)+t_i}^{t+t_i} p(\xi_) d\xi - \int_{\sigma(t_i}^{t_i} p(\xi_) d\xi$$

$$+ \int_{\sigma(t_i}^{t_i} p(\xi_) d\xi$$

$$\int_{\sigma(ti)}^{t_i} p(\xi)d\xi \geq \frac{1+\varepsilon}{e} - \left(\int_{\sigma(t_i}^{t_i} p(\xi_) d\xi - \int_{\sigma(t)+t_i}^{t+t_i} p(\xi_) d\xi \right).$$

Hence

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s+t_i)ds \geq \frac{1+\varepsilon}{e}$$

$$- \liminf_{t\to+\infty} \left(\int_{\sigma(t_i}^{t_i} p(\xi_) d\xi - \int_{\sigma(t)+t_i}^{t+t_i} p(\xi_) d\xi \right).$$

According to (2.15), for large $c = i_0 \in \{1, 2, \dots\}$ we have

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(\xi+i_0)d\xi > \frac{1}{e}\,.$$

Therefore by Theorem 2.2, inequality (2.13) is oscillatory. The obtained contradiction proves that equation (1.1) is oscillatory.

Remark 2.2. The sufficiently of (2.16) for equation (1.1) to be oscillatory is proved in [1] under the real additional conditions of p being bounded and σ being uniform conditions in neighbourhood of $+\infty$.

R E F E R E N C E S

1. Koplatadze R. Zeros os solutions of first-order differential equations with retarded argument (Russian). *Tbilisi Gos. Univer. Inst. Prikl. Math. Trudy*, **4** (1983), 128-134.

2. Myshkis A. Linear differential equations with retarded argument (Russian). *Nauka, Moscow*, 1972.

3. Koplatadze R., Chanturia T. On oscillatory properties of differential equations with a deviating argument (Russian). *Izdat. Tbilis. Univ., Tbilisi*, 1977, 115pp.

4. Koplatadze R. On oscillatory properties of solutions of functional differential equations (Russian). *Mem. Differential Equations, Math. Phys*, **3** (1984), 1-177.

5. Koplatadze R., Chanturia T. On oscillatory and monitor solution of first order differential equation with retarded argument (Russia). *Differential'nye Uravnenia*, **18**, 8 (1982), 1463-1465.

6. Garab A., Pituk M., Stavroulakis I. A sharp oscillation for a linear differential equation. *Applied Mathematics Letters*, **23** (2019), 58-65.

7. Garab A., Stavroulakis I. Oscillation criteria for first order linear delay differential equations. *Applied Mathematics Letter* (online 4/april), 2020.

8. Infante G., Koplatadze R., Stavroulakis I. Oscillation criteria for differential equations with several retarded arguments. **58**, 3 (2015), 347-364.

9. Koplatadze R. Specific properties of solution of first order differential equations with several delay argument. *J. Contemporary Math. Anal.*, **50**, 5 (2015), 229-235.

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