Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 50, 2024

REPRESENTATION FORMULAS OF SOLUTION FOR A PERTURBED CONTROLLED FUNCTIONAL-DIFFERENTIAL EQUATION CONSIDERING VARIATION OF THE INITIAL MOMENT AND CONTINUITY OF THE INITIAL CONDITION

Berdawood K., Berikelashvili G., Dvalishvili Ph., Tadumadze T.

Abstract. The analytic representation formulas of solution are proved for the nonlinear perturbed controlled functional-differential equation with a delay parameter. In formulas the effects of the continuous initial condition and variation of the initial moment are revealed. The representation formula of solution plays an important role in the investigation of optimization problems, allows one to get an approximate solution of the perturbed equation and to carry out a sensitivity analysis of mathematical models.

Keywords and phrases: Controlled functional-differential equation, represent-ation formula of solution, continuous initial condition, perturbations.

AMS subject classification (2020): 4K27, 34K07.

1. Introduction

In the present paper is found an analytic relation (representation formula) between solution of the original Cauchy problem

$$\dot{x}(t) = f(t, x(t), x(t-\tau_0), u_0(t), u_0(t-\theta)), t \in [t_{00}, t_1], \tag{1.1}$$

$$x(t) = \varphi_0(t), t \le t_{00}$$

and solution of corresponding perturbed (with respect to initial moment t_{00}, delay τ_0, initial function $\varphi_0(t)$ and control function $u_0(t)$) problem are found. Continuity at the initial moment means that at the initial moment values of the initial function and trajectory always coincide. The representation formula of solution plays an important role in the investigation of optimization problems [1-8] and in carring out a sensitivity analysis of mathematical Models [9-11]. Finally, we note that the case when equation (1.1) does not contain $u_0(t-\theta)$ and the initial moment is fixed the representation formulas of solution are proved in [12, 13].

2. Formulation of main results

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$ and let $O \subset \mathbb{R}^n$, $U \subset \mathbb{R}^r$ be convex, open and bounded sets; let $t_1 > t_{02} > t_{01}$, $\tau_2 > \tau_1 > 0$ and let $\theta > 0$ be given numbers, with $t_{02} + \tau_2 < t_1$. Suppose that the n-dimensional function $f(t, x, y, u, v)$ is continuous on the set $I \times O^2 \times U^2$, where $I = [t_{01}, t_1]$ and continuously differentiable with respect to x, y, u, v. It is clear that, for the compact sets $K_0 \subset O$ and $U_0 \subset U$ there exists a number $M_0 = M_0(K_0, U_0) > 0$ such that

$$|f(t, x, y, u, v)| + |f_x(\cdot)| + |f_y(\cdot)| + |f_u(\cdot)| + |f_v(\cdot)| \le M_0, \tag{2.1}$$

$$\forall (t, x, y, u, v) \in I \times K_0^2 \times U_0^2.$$

Further, denote by Φ and Ω the sets of continuous differentiable functions $\varphi(t) \in O, t \in [\hat{\tau}, t_{02}]$, where $\hat{\tau} = t_{01} - \max\{\tau_2, \theta\}$ and piecewise continuous functions $u(t) \in U, t \in I_1 = [\hat{\tau}, t_1]$, respectively, with the set $clu(I_1) \subset U$.

To each element

$$w = (t_0, \tau, \varphi(t), u(t)) \in W = (t_{01}, t_{02}) \times (\tau_1, \tau_2) \times \Phi \times \Omega$$

we assign the controlled functional-differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t), u(t-\theta)), t \in [t_0, t_1] \tag{2.2}$$

with the continuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{2.3}$$

Definition 2.1. Let $w \in W$, a function $x(t) = x(t; w) \in O, t \in I_1$ is called a solution of equation (2.2) with the condition (2.3) or a solution corresponding to the element w and defined on the interval I_1 if $x(t)$ satisfies condition (2.3) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.2) almost everywhere (a. e.) on $[t_0, t_1]$.

Let us introduce the notations:

$$|w| = |t_0| + |\tau| + \|\varphi\|_1 + \|u\|, \ W_\varepsilon(w_0) = \Big\{ w \in W : |w - w_0| \le \varepsilon \Big\},$$

where

$$\|\varphi\|_1 = \sup \Big\{ |\varphi(t)| + |\dot{\varphi}(t)| : t \in [\hat{\tau}, t_{02}] \Big\}, \|u\| = \sup \Big\{ |u(t)| : t \in I_1 \Big\},$$

$\varepsilon > 0$ is a fixed number and $w_0 = (t_{00}, \tau_0, \varphi_0(t), u_0(t)) \in W$ is a fixed element; furthermore,

$$\delta t_0 = t_0 - t_{00}, \ \delta\tau = \tau - \tau_0, \ \delta\varphi(t) = \varphi(t) - \varphi_0(t), \ \delta u(t) = u(t) - u_0(t),$$
$$\delta w = w - w_0 = (\delta t_0, \delta\tau, \delta\varphi(t), \delta u(t)), |\delta w| = |\delta t_0| + |\delta\tau| + \|\delta\varphi\|_1 + \|\delta u\|.$$

Let $x_0(t) = x(t; w_0)$ be a solution corresponding to the element $w_0 = (t_{00}, \tau_0, \varphi_0(t), u_0(t)) \in W$ and defined on the interval I_1. Then there exists a number $\varepsilon_1 > 0$ such that to each element $w = w_0 + \delta w \in W_{\varepsilon_1}(w_0)$ corresponds the solution $x(t; w_0 + \delta w)$ defined on the interval I_1 (see Theorem 3.1, in Section 3).

Theorem 2.1. *Let $x_0(t) = x(t; w_0)$ be a solution corresponding to the element $w_0 = (t_{00}, \tau_0, \varphi_0(t), u_0(t)) \in W$ and defined on the interval I_1. Then there exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for arbitrary*

$$\delta w \in \delta W^-_{\varepsilon_2} = \Big\{ \delta w \in W - w_0 : \ |\delta w| \le \varepsilon_2, \ \delta t_0 \le 0 \Big\}$$

on the interval $[t_{00}, t_1]$ the following representation holds:

$$x(t; w_0 + \delta w) = x_0(t) + \delta x(t; \delta w) + o(t; \delta w), \tag{2.4}$$

where

$$\delta x(t;\delta w)=Y(t_{00};t)\Big(\dot{\varphi}_0(t_{00})-f_0^-\Big)\delta t_0+\beta(t;\delta w) \tag{2.5}$$

and

$$\beta(t;\delta w)=Y(t_{00};t)\delta\varphi(t_{00})+\int_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)ds$$

$$-\Big\{\int_{t_{00}}^{t}Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau$$

$$+\int_{t_{00}}^{t}Y(s;t)\Big[f_u[s]\delta u(s)+f_v[s]\delta u(s-\theta)\Big]ds. \tag{2.6}$$

Here

$$\lim_{|\delta w|\to 0}\frac{o(t;\delta w)}{|\delta w|}=0 \text{ uniformly for } t\in[t_{00},t_1];$$

$$f_0^-=f(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0),u_0(t_{00}-),u_0(t_{00}-\theta-)), \tag{2.7}$$

$$f_u[s]=\frac{\partial}{\partial u}f(s,x_0(s),x_0(s-\tau_0),u_0(s),u_0(s-\theta));$$

$Y(s;t)$ *is the* $n\times n$ *matrix function satisfying the equation*

$$Y_s(s;t)=-Y(s;t)f_x[s]-Y(s+\tau_0;t)f_y[s+\tau_0],\, s\in[t_{00},t],\ t\in(t_{00},t_1] \tag{2.8}$$

and the conditions

$$Y(t;t)=E;\ Y(s;t)=\Theta,\ s>t; \tag{2.9}$$

E *is the identity matrix and* Θ *is the zero matrix.*

Theorem 2.1 corresponds to the case when the variation at the initial moment t_{00} occurs from the left.

Some comments. The function $\delta x(t;\delta w)$ is called the first variation of the solution $x_0(t)$ on the interval $[t_{00},t_1]$. The expression (2.5) is called the variation formula of solution. The term "variation formula of solution" has been introduced by Revaz Gamkrelidze and proved for the ordinary differential equation in [1].

The expression

$$Y(t_{00};t)\Big(\dot{\varphi}_0(t_{00})-f_0^-\Big)\delta t_0$$

in formula (2.5) is the effect of perturbation of the moment t_{00} and the continuous initial condition.

The addend

$$Y(t_{00};t)\delta\varphi(t_{00})+\int_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)ds$$

in formula (2.6) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression

$$-\Big\{\int_{t_{00}}^{t}Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau$$

in formula (2.6) is the effect of perturbation of the delay parameter τ_0.

The addend

$$\int_{t_0}^{t}Y(s;t)\Big[f_u[s]\delta u(s)+f_v[s]\delta u(s-\theta)\Big]ds$$

in formula (2.6) is the effect of perturbation of the control function $u_0(t)$.

On the basis of the Cauchy formula [6] we conclude that the function

$$\delta x(t)=\begin{cases}\delta\varphi(t),\ t\in[\hat{\tau},t_{00}),\\(\dot{\varphi}_0(t_{00})-f_0^-)\delta t_0+\delta\varphi(t_{00}),t=t_{00},\\\delta x(t;\delta w),t\in[t_{00},t_1]\end{cases}$$

satisfies the linear functional-differential equation

$$\dot{\delta x}(t)=f_x[t]\delta x(t)+f_y[t]\delta x(t-\tau_0)-f_y[t]\dot{x}_0(t-\tau_0)\delta\tau$$

$$+f_u[t]\delta u(t)+f_v[t]\delta u(t-\theta),t\in(t_{00},t_1] \qquad (2.10)$$

with the initial condition

$$\delta x(t)=\delta\varphi(t),t\in[\hat{\tau},t_{00}),\delta x(t_{00})=(\dot{\varphi}_0(t_{00})-f_0^-)\delta t_0+\delta\varphi(t_{00}). \qquad (2.11)$$

Formula (2.4) allows us to construct on the interval $[t_{00},t_1]$ an approximate solution of the perturbed equation

$$\dot{x}(t)=f(t,x(t),x(t-\tau_0-\delta\tau),u_0(t)+\delta u(t),u_0(t-\theta)+\delta u(t-\theta)) \qquad (2.12)$$

with the initial condition

$$x(t)=\varphi_0(t)+\delta\varphi(t),t\in[\hat{\tau},t_{00}+\delta t_0], \qquad (2.13)$$

where $\delta t_0<0$.

In fact, for a small $|\delta w|$ from (2.4) for solution $x(t;w_0+\delta w)$ of the perturbed problem (2.12)-(2.13) we have

$$x(t;w_0+\delta w)\approx x_0(t)+\delta x(t;\delta w),t\in[t_{00},t_1].$$

Thus, $x_0(t) + \delta x(t; \delta w)$ can be considered as approximate solution on the interval $[t_{00}, t_1]$. It is clear that, the first variation $\delta x(t; \delta w)$ can be calculated in two ways: first -by finding the solution $Y(s; t)$ of problem (2.8)-(2.9); the other- by finding the solution of the problem (2.10)-(2.11).

Theorem 2.2. *Let $x_0(t) = x(t; w_0)$ be a solution corresponding to the element $w_0 = (t_{00}, \tau_0, \varphi_0(t), u_0(t)) \in W$ and defined on the interval I_1. Then for each fixed $\hat{t}_0 \in (t_{00}, t_{00}+\delta)$, where $\delta > 0$ and $t_{00} + \delta < t_{02}$ there exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for arbitrary*

$$\delta w \in \delta W^+_{\varepsilon_2} = \Big\{\delta w \in W - w_0 : \ |\delta w| \le \varepsilon_2, \ \delta t_0 > 0\Big\}$$

on the interval $[\hat{t}_0, t_1]$ the representation (2.4) holds, where

$$\delta x(t; \delta w) = Y(t_{00}; t)\Big(\dot{\varphi}_0(t_{00}) - f^+_0\Big)\delta t_0 + \beta(t; \delta w). \tag{2.14}$$

Theorem 2.2 corresponds to the case when the variation at the point t_{00} occurs from the right.

Theorem 2.3. *Let $x_0(t) = x(t; w_0)$ be a solution corresponding to the element $w_0 = (t_{00}, \tau_0, \varphi_0(t), u_0(t)) \in W$ and defined on the interval I_1. Moreover, let*

$$f^+_0 = f^-_0 := f_0.$$

Then for each fixed $\hat{t}_0 \in (t_{00}, t_{00} + \delta)$, where $\delta > 0$ and $t_{00} + \delta < t_{02}$ there exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for arbitrary

$$\delta w \in \delta W_{\varepsilon_2} = \Big\{\delta w \in W - w_0 : \ |\delta w| \le \varepsilon_2\Big\}$$

on the interval $[\hat{t}_0, t_1]$ the representation (2.4) holds, where

$$\delta x(t; \delta w) = Y(t_{00}; t)\Big(\dot{\varphi}_0(t_{00}) - f_0\Big)\delta t_0 + \beta(t; \delta w).$$

Theorem 2.3 corresponds to the case when the variation at point t_{00} occurs from both sides and is a corollary to Theorems 2.1 and 2.2.

3. Auxiliary assertions

Theorem 3.1 ([6], p. 18]). *Let $x_0(t) = x(t; w_0)$ be a solution corresponding to the element $w_0 = (t_{00}, \tau_0, \varphi_0(t), u_0(t)) \in W$ and defined on the interval $I_1 = [\hat{\tau}, t_1]$. Then there exists a number $\varepsilon_1 > 0$ such that to each element $w = w_0 + \delta w \in W_{\varepsilon_1}(w_0)$ corresponds solution $x(t) = x(t; w_0 + \delta w)$ defined on the interval I_1 with $x(t) \in K_1$ and $u_0(t) + \delta u(t) \in U_1$, where $K_1 \subset O$ is a compact set containing a neighborhood of the set $x_0(I_1)$ and $U_1 \subset U$ is a compact set containing a neighborhood of the set $clu_0(I_1)$.*

Theorem 3.1 allows one to introduce the increment of the solution $x_0(t)$ on the interval I_1

$$\Delta x(t) = \Delta x(t; \delta w) = x(t; w_0 + \delta w) - x_0(t),$$
$$t \in I_1, \delta w = w - w_0 \in \delta W^-_{\varepsilon_1}.$$

Theorem 3.2. *There exists a number $\varepsilon_2 \in (0, \varepsilon_1)$ such that*

$$\max_{t \in I_1} |\Delta x(t)| \leq O(\delta w) \tag{3.1}$$

for arbitrary $\delta w \in \delta W^-_{\varepsilon_2}$. Moreover,

$$\Delta x(t_{00}) = \delta\varphi(t_{00}) + \Big(\dot{\varphi}_0(t_{00}) - f^-_0\Big)\delta t_0 + o(\delta w). \tag{3.2}$$

Here

$$\lim_{|\delta w| \to 0} \frac{O(\delta w)}{|\delta w|} < \infty.$$

Proof. Let $\varepsilon_2 \in (0, \varepsilon_1)$ be insomuch small that for arbitrary $\delta w \in \delta W^-_{\varepsilon_2}$ the following inequality

$$t_{00} - \tau < t_0, \ t_{00} - \tau_0 < t_0 \tag{3.3}$$

holds, where $\tau = \tau_0 + \delta\tau$ and $t_0 = t_{00} + \delta t_0$. On the interval $[t_{00}, t_1]$ the function $\Delta x(t) = x(t) - x_0(t)$, where $x(t) = x(t; w_0 + \delta w)$, satisfies the equation

$$\dot{\Delta} x(t) = a(t; \delta w), \tag{3.4}$$

where

$$a(t; \delta w) = f(t, x(t), x(t-\tau), u(t), u(t-\theta)) - f(t, x_0(t), x_0(t-\tau_0), u_0(t), u_0(t-\theta)),$$

$$u(t) = u_0(t) + \delta u(t).$$

We rewrite the equation (3.4) in the integral form

$$\Delta x(t) = \Delta x(t_{00}) + \int_{t_{00}}^{t} a(s; \delta w) ds, t \in [t_{00}, t_1].$$

Hence it follows that

$$|\Delta x(t)| \leq |\Delta x(t_{00})| + a_1(t; t_{00}, \delta w), \tag{3.5}$$

where

$$a_1(t; t_{00}, \delta w) = \int_{t_{00}}^{t} |a(s; \delta w)| ds.$$

Let us prove the formula (3.2). We have

$$\Delta x(t_{00}) = x(t_{00}) - x_0(t_{00}) = \varphi_0(t_0) + \delta\varphi(t_0)$$

$$+ \int_{t_0}^{t_{00}} f(t, x(t), \varphi(t-\tau), u(t), u(t-\theta)) dt - \varphi_0(t_{00}) \tag{3.6}$$

(see (3.3)). It is clear that

$$\varphi_0(t_0) - \varphi_0(t_{00}) = \int_{t_{00}}^{t_0} \dot{\varphi}_0(t)dt = \dot{\varphi}_0(t_{00})\delta t_0 + \alpha(\delta w),$$

where

$$\alpha(\delta w) = \int_{t_0}^{t_{00}} \Big(\dot{\varphi}_0(t) - \dot{\varphi}_0(t_{00})\Big)dt$$

and

$$\delta\varphi(t_0) = \delta\varphi(t_{00}) + \delta\varphi(t_0) - \delta\varphi(t_{00}) = \delta\varphi(t_{00}) + \beta(\delta w),$$

where

$$\beta(\delta w) = \int_{t_0}^{t_{00}} \dot{\delta}\varphi(t)dt.$$

It is easy to see that

$$|\alpha(\delta w)| \le |\delta w| \max_{t\in[t_0,t_{00}]} |\dot{\varphi}_0(t) - \dot{\varphi}_0(t_{00})| = o(\delta w)$$

and

$$|\beta(\delta w)| \le |\delta w|^2 = o(\delta w).$$

Thus,

$$\varphi_0(t_0) + \delta\varphi(t_0) - \varphi_0(t_{00}) = \dot{\varphi}(t_{00})\delta t_0 + \delta\varphi(t_{00}) + o(\delta w). \tag{3.7}$$

Furthermore,

$$\int_{t_0}^{t_{00}} f(t, x(t), \varphi(t-\tau), u(t), u(t-\theta))dt = \gamma_1(\delta w) + \gamma_2(\delta w),$$

where

$$\gamma_1(\delta w) = \int_{t_0}^{t_{00}} f(t, \varphi_0(t), \varphi_0(t-\tau_0), u_0(t), u_0(t-\theta))dt,$$

and

$$\gamma_2(\delta w) = \int_{t_0}^{t_{00}} \Big[f(t, x(t), \varphi(t-\tau), u(t), u(t-\theta)) \\ - f(t, \varphi_0(t), \varphi_0(t-\tau_0), u_0(t), u_0(t-\theta))\Big]dt.$$

On the basis of (2.1) it is proved that for the compact sets $K_1 \subset O$ and $U_1 \subset U$ there exist a number $L > 0$ such that

$$|f(t,x_1,y_1,u_1,v_1)-f(t,x_2,y_2,u_2,v_2)|\le L\Big(|x_1-x_2|+|y_1-y_2|+|u_1-u_2|$$
$$+|v_1-v_2|\Big),\forall t\in[t_{01},t_1],\forall(x_1,x_2,y_1,y_2)\in K_1^4,\forall(u_1,u_2,v_1,v_2)\in U_1^4$$

(see [6], p. 29).

Let us now transform $\gamma_1(\delta w)$ and $\gamma_2(\delta w)$. We have

$$\gamma_1(\delta w)=\int_{t_0}^{t_{00}}f(t,\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0),u_0(t_{00}-),u_0(t_{00}-\theta-))dt$$
$$+\int_{t_0}^{t_{00}}\Big[f(t,\varphi(t),\varphi(t-\tau),u(t),u(t-\theta))$$
$$-f(t,\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0),u_0(t_{00}-),u_0(t_{00}-\theta-))\Big]dt$$
$$=-f_0^-\delta t_0+\gamma_{11}(\delta w)$$

(see (2.7)). Next,

$$|\gamma_{11}(\delta w)|\le L\int_{t_0}^{t_{00}}\Big[|\varphi(t)-\varphi_0(t_{00})|+|\varphi(t-\tau)-\varphi_0(t_{00}-\tau_0)|$$
$$+|u(t)-u_0(t_{00}-)|+|u(t-\theta)-u_0(t_{00}-\theta-)|\Big]dt\le L|\delta w|\gamma_{12}(\delta w),$$

where

$$\gamma_{12}(\delta w)=\sup_{t\in[t_0,t_{00}]}\Big[|\varphi(t)-\varphi_0(t_{00})|+|\varphi(t-\tau)-\varphi_0(t_{00}-\tau_0)|$$
$$+|u(t)-u_0(t_{00}-)|+|u(t-\theta)-u_0(t_{00}-\theta-)|\Big].$$

It is clear that

$$\gamma_{12}(\delta w)\to 0 \text{ for } |\delta w|\to 0,$$

i. e.,

$$\gamma_{11}(\delta w)=o(\delta w).$$

Thus,

$$\gamma_1(\delta w)=-f_0^-\delta t_0+o(\delta w). \tag{3.8}$$

It is easy to see that

$$|\gamma_2(\delta w)|\le L\int_{t_0}^{t_{00}}\Big(|x(t)-\varphi_0(t)|+|\varphi_0(t-\tau)+\delta\varphi(t-\tau)-\varphi_0(t-\tau_0)|$$
$$+|\delta u(t)|+|\delta u(t-\theta)|\Big)dt=L\int_{t_0}^{t_{00}}|x(t)-\varphi_0(t)|dt+o(\delta w);$$

For $t \in [t_0, t_{00}]$ we have

$$|x(t) - \varphi_0(t)| = |x(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau), u(s), u(s-\theta))ds - \varphi_0(t)|$$

$$= |\varphi_0(t_0) + \delta\varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau), u(s), u(s-\theta))ds - \varphi_0(t)|$$

$$\leq \Big\{|\delta\varphi(t_0)| + \int_{t_0}^{t_{00}} |f(s, x(s), x(s-\tau), u(s), u(s-\theta))|ds$$

$$+ \int_{t_0}^{t_{00}} |\dot{\varphi}_0(s)|ds\Big\} \leq O(\delta w) \quad (3.9)$$

(see Theorem 3.1).
Consequently,

$$\gamma_2(\delta w) = o(\delta w) \quad (3.10).$$

From (3.6) by virtue of (3.7), (3.8) and (3.10) we obtain (3.2).

Let us now prove the inequality (3.1). We note that $\delta w \in \delta W_{\varepsilon_2}^{-}$, i. e., $t_0 < t_{00}$, therefore

$$\Delta x(t) = \begin{cases} \delta\varphi(t), t \in [\hat{\tau}, t_0), \\ x(t) - \varphi_0(t), [t_0, t_{00}]. \end{cases}$$

On the basis of (3.9) we have

$$\max_{t \in [\hat{\tau}, t_{00}]} |\Delta x(t)| \leq O(\delta w). \quad (3.11)$$

We will estimate now $a_1(t; t_{00}, \delta w), t \in [t_{00}, t_1]$. We have

$$a_1(t; t_{00}, \delta w) \leq L \int_{t_{00}}^{t} \Big[|x(s) - x_0(s)| + |x(s-\tau) - x_0(s-\tau_0)|$$

$$+|\delta u(s)| + |\delta u(s-\theta)|\Big]ds \leq L \int_{t_{00}}^{t} |\Delta x(s)|ds + La_{11}(t; t_{00}, \delta w) + 2L(t_1 - t_{00})|\delta w|$$

$$\leq O(\delta w) + L \int_{t_{00}}^{t} |\Delta x(s)|ds + La_{11}(t; t_{00}, \delta w),$$

where

$$a_{11}(t; t_{00}, \delta w) = \int_{t_{00}}^{t} |x(s-\tau) - x_0(s-\tau_0)|ds.$$

We introduce the notations

$$s_1 = \min\{t_0 + \tau, t_0 + \tau_0\}, s_2 = \max\{t_{00} + \tau, t_{00} + \tau_0\}.$$

It is not difficult to see that

$$s_1 > t_{00} \text{ and } |s_2 - s_1| = O(\delta w)$$

(see (3.3)). Next, if $t \in [t_{00}, s_1]$ then $t - \tau \le t_0$ and $t - \tau_0 \le t_0 \le t_{00}$; if $t \in [s_2, t_1]$ then $t - \tau \ge t_{00}$ and $t - \tau_0 \ge t_{00}$; there exist numbers $M > 0$ and $N > 0$ such that

$$|x(s-\tau) - x_0(s-\tau_0)| \le M,\ t \in [t_{00}, t_1]$$

and

$$|\dot{x}_0(t)| \le N, \text{ a. e. on } I_1$$

(see Theorem 3.1, Definition 2.1 and (2.1)).
Let $t \in [t_{00}, s_1]$ then

$$a_{11}(t; t_{00}, \delta w) \le a_{11}(s_1; t_{00}, \delta w) = \int_{t_{00}}^{s_1} |\varphi(s-\tau) - \varphi_0(s-\tau_0)| ds$$
$$\le \int_{t_{00}}^{s_1} |\delta\varphi(s-\tau)| ds + \int_{t_{00}}^{s_1} |\varphi_0(s-\tau) - \varphi_0(s-\tau_0)| ds \le (t_1 - t_{00})|\delta w|$$
$$+ \int_{t_{00}}^{s_1} \Big| \int_{s-\tau_0}^{s-\tau} |\dot{\varphi}_0(\xi)| d\xi \Big| ds \le (t_1 - t_{00})|\delta w| + (t_1 - t_{00})\|\varphi_0\|_1 |\delta w| = O(\delta w).$$

Let $t \in [s_1, s_2]$ then

$$a_{11}(t; t_{00}, \delta w) = a_{11}(s_1; t_{00}, \delta w) + \int_{s_1}^{t} |x(s-\tau) - x_0(s-\tau_0)| ds$$
$$\le O(\delta w) + M|s_2 - s_1| = O(\delta w).$$

Let $t \in [s_2, t_1]$ then $s_2 - \tau \ge t_{00}$ and $s_2 - \tau_0 \ge t_{00}$. We have

$$a_{11}(t; t_{00}, \delta w) = a_{11}(s_2; t_{00}, \delta w) + \int_{s_2}^{t} |x(s-\tau) - x_0(s-\tau_0)| ds$$
$$\le O(\delta w) + \int_{s_2}^{t} |x(s-\tau) - x_0(s-\tau)| ds + \int_{s_2}^{t} |x_0(s-\tau) - x_0(s-\tau_0)| ds$$
$$= O(\delta w) + \int_{s_2}^{t} |\Delta x(s-\tau)| ds + \int_{s_2}^{t} \Big| \int_{s-\tau_0}^{s-\tau} |\dot{x}_0(\xi)| d\xi \Big| ds$$
$$\le O(\delta w) + \int_{s_{2-\tau}}^{t-\tau} |\Delta x(s)| ds + (t - s_2) N |\delta\tau|$$
$$\le O(\delta w) + \int_{t_{00}}^{t} |\Delta x(s)| ds + (t_1 - t_{00}) N |\delta w|$$
$$= O(\delta w) + \int_{t_{00}}^{t} |\Delta x(s)| ds.$$

By using the last relations we obtain

$$a_{11}(t;t_{00},\delta w)\leq O(\delta w)+\int_{t_{00}}^{t}|\Delta x(s)|ds, t\in[t_{00},t_1].$$

Consequently, we get

$$a_1(t;t_{00},\delta w)\leq O(\delta w)+2\int_{t_{00}}^{t}|\Delta x(s)|ds. \tag{3.12}$$

From (3.5) according to (3.2) and (3.12) we have

$$|\Delta x(t)|\leq O(\delta w)+2\int_{t_{00}}^{t}|\Delta x(s)|ds, t\in[t_{00},t_1].$$

By the Gronwall-Bellman inequality we get

$$|\Delta x(t)|\leq O(\delta w)e^{2L(t_1-t_{00})}=O(\delta w), t\in[t_{00},t_1]. \tag{3.13}$$

According to (3.11) and (3.13) we obtain (3.1).

Theorem 3.3. *There exists a number $\varepsilon_2\in(0,\varepsilon_1)$ such that*

$$\max_{t\in I_1}|\Delta x(t)|\leq O(\delta w)$$

for arbitrary $\delta w\in\delta W^+_{\varepsilon_2}$. Moreover,

$$\Delta x(t_{00})=\delta\varphi(t_{00})+\Big(\dot\varphi_0(t_{00})-f_0^+\Big)\delta t_0+o(\delta w).$$

Theorem 3.3 is proved by analogy to Theorem 3.2 without principal changes.

4. Proof of Theorem 2.1

The function $\Delta x(t)$ satisfies the equation

$$\dot\Delta x(t)=f_x[t]\Delta x(t)+f_y[t]\Delta x(t-\tau_0)+f_u[t]\delta u(t)+f_v[t]\delta u(t-\theta)$$
$$+b(t;\delta w), t\in[t_{00},t_1], \tag{4.1}$$

where

$$b(t;\delta w)=a(t;\delta w)-f_x[t]\Delta x(t)-f_y[t]\Delta x(t-\tau_0)-f_u[t]\delta u(t)$$
$$-f_v[t]\delta u(t-\theta). \tag{4.2}$$

By using the Cauchy formula [6, p. 31], one can represent the solution of the equation (4.1) in the form

$$\Delta x(t)=Y(t_{00};t)\Delta x(t_{00})+\int_{t_{00}}^{t}Y(s;t)\Big(f_u[s]\delta u(s)+f_v[s]\delta u(s-\theta)\Big)ds$$
$$+b_1(t;t_{00},\delta w)+b_2(t;t_{00},\delta w), \tag{4.3}$$

where

$$b_1(t;t_{00},\delta w)=\int_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\Delta x(s)ds,$$

$$b_2(t;t_{00},\delta w)=\int_{t_{00}}^{t}Y(s;t)b(s;\delta w)ds$$

and $Y(s;t)$ is the matrix function satisfying the equation (2.8) and the condition (2.9). The function $Y(\xi;t)$ is continuous on the set

$$\Pi=\Big\{(s,t):s\in[t_{00},t],\ t\in[t_{00},t_1]\Big\}$$

(see [6], Lemma 2.6). Therefore,

$$Y(t_{00};t)\Delta x(t_{00})=Y(t_{00};t)\Big[\delta\varphi(t_{00})+\Big(\dot\varphi_0(t_{00})-f_0^-\Big)\delta t_0\Big]+o(t;\delta w) \tag{4.4}$$

(see (3.2)). One can readily see that

$$b_1(t;t_{00},\delta w)=\int_{t_{00}-\tau_0}^{t_0}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)ds+\int_{t_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\Delta x(s)ds$$

$$=\int_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)ds-\int_{t_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)ds$$

$$+\int_{t_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\Delta x(s)ds=\int_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)ds$$

$$+o(t;\delta w) \tag{4.5}$$

(see (3.1)). We introduce the notations:

$$f[t;s,\delta w]=f(t,x_0(t)+s\Delta x(t),x_0(t-\tau_0)+s(x_0(t-\tau)-x_0(t-\tau_0)$$

$$+\Delta x(t-\tau)),u_0(t)+s\delta u(t),u_0(t-\theta)+s\delta u(t-\theta))$$

$$\sigma_x(t;s,\delta w)=f_x[t;s,\delta w]-f_x[t].$$

It is easy to see that

$$a(t;\delta w)=\int_0^1\frac{d}{ds}f[t;s,\delta w]ds=\int_0^1\Big\{f_x[t;s,\delta w]\Delta x(t)$$

$$+f_y[t;s,\delta w](x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau))+f_u[t;s,\delta w]\delta u(t)$$

$$+f_v[t;s,\delta w]\delta u(t-\theta)\Big\}ds=\Big[\int_0^1\sigma_x(t;s,\delta w)ds\Big]\Delta x(t)$$

$$+\Big[\int_0^1\sigma_y(t;s,\delta w)ds\Big](x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau))$$

$$+\Big[\int_0^1 \sigma_u(t;s,\delta w)ds\Big]\delta u(t)+\Big[\int_0^1 \sigma_v(t;s,\delta w)ds\Big]\delta u(t-\theta)$$
$$+f_x[t]\Delta x(t)+f_y[t](x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau))+f_u[t]\delta u(t)$$
$$+f_v[t]\delta u(t-\theta).$$

Taking into account the last relation for $t\in[t_{00},t_1]$, we have

$$b_2(t;t_{00},\delta w)=b_{21}(t;\delta w)+b_{22}(t;\delta w)+b_{23}(t;\delta w)+b_{24}(t;\delta w)$$
$$+b_{25}(t;\delta w)+b_{26}(t;\delta w),$$

where

$$b_{21}(t;\delta w)=\int_{t_{00}}^t Y(\xi;t)\sigma_x(\xi;\delta w)\Delta x(\xi)d\xi, \sigma_x(\xi;\delta w)=\int_0^1 \sigma_x(\xi;s,\delta w)ds;$$
$$b_{22}(t;\delta w)=\int_{t_{00}}^t Y(\xi;t)\sigma_y(\xi;\delta w)(x_0(\xi-\tau)-x_0(\xi-\tau_0)+\Delta x(\xi-\tau))d\xi,$$
$$\sigma_y(\xi;\delta w)=\int_0^1 \sigma_y(\xi;s,\delta w)ds; b_{23}(t;\delta w)=\int_{t_{00}}^t Y(\xi;t)\sigma_u(\xi;\delta w)\delta u(\xi)d\xi,$$
$$\sigma_u(\xi;\delta w)=\int_0^1 \sigma_u(\xi;s,\delta w)ds; b_{24}(t;\delta w)=\int_{t_{00}}^t Y(\xi;t)\sigma_v(\xi;\delta w)\delta u(\xi-\theta)d\xi,$$
$$\sigma_v(\xi;\delta w)=\int_0^1 \sigma_v(\xi;s,\delta w)ds; b_{25}(t;\delta w)=\int_{t_{00}}^t Y(\xi;t)f_y[\xi][x_0(\xi-\tau)-x_0(\xi-\tau_0)]d\xi;$$
$$b_{26}(t;\delta w)=\int_{t_{00}}^t Y(\xi;t)f_y[\xi][\Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)]d\xi$$

(see 4.2). The function $x_0(t), t\in I_1$, is absolutely continuous. For each Lebesgue point $\xi\in(t_{00},t_1]$ of the function $\dot x_0(\xi-\tau_0)$ we get

$$x_0(\xi-\tau)-x_0(\xi-\tau_0)=\int_{\xi}^{\xi-\delta\tau}\dot x_0(\varsigma-\tau_0)d\varsigma=$$
$$=-\dot x_0(\xi-\tau_0)\delta\tau+\gamma(\xi;\delta\tau), \tag{4.6}$$

and with

$$\lim_{|\delta\tau|\to 0}\Big|\frac{\gamma(\xi;\delta\tau)}{\delta\tau}\Big|=0.$$

We denote now $\gamma(\xi;\delta\tau)$ by $\gamma(\xi;\delta w)$. It is clear that

$$\lim_{|\delta w|\to 0}\frac{|\gamma(\xi;\delta w|}{|\delta w|}\le \lim_{|\delta\tau|\to 0}\Big|\frac{\gamma(\xi;\delta\tau)}{\delta\tau}\Big|=0. \tag{4.7}$$

Thus, (4.6) is valid for almost all points of the interval (t_{00},t_1). From (4.6), taking into account the boundedness of the function $\dot x_0(t), t\in I_1$ it follows that

$$|x_0(\xi-\tau)-x_0(\xi-\tau_0)|\le N|\delta\tau|\le N|\delta w|=O(\delta w) \tag{4.8}$$

and

$$\frac{|\gamma(\xi;\delta w|}{|\delta w|}\le\Big|\frac{\gamma(\xi;\delta\tau)}{\delta\tau}\Big|=\Big|\dot{x}_0(\xi-\tau_0)+\frac{1}{\delta\tau}\int_{\xi}^{\xi-\delta\tau}\dot{x}_0(\varsigma-\tau_0)d\varsigma\Big|\le const. \tag{4.9}$$

It is clear that for $\xi\in[t_{00},s_1]$

$$\begin{aligned}|\Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)|&=|\delta\varphi(\xi-\tau)-\delta\varphi(\xi-\tau_0)|\\&\le\Big|\int_{\xi-\tau_0}^{\xi-\tau}\dot{\delta}\varphi(\varsigma)d\varsigma\Big|=o(\delta w)\end{aligned} \tag{4.10}$$

and for $\xi\in[s_1,s_2]$ we have

$$|\Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)|\le O(\delta w) \tag{4.11}$$

(see (3.1)).

Let $\xi\in[s_2,t_1]$ then $\xi-\tau\ge t_{00},\xi-\tau_0\ge t_{00}$. Therefor,

$$|\Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)|=\Big|\int_{\xi-\tau}^{\xi-\tau_0}|\dot{\Delta}(s)|ds\Big|$$

$$\le\Big|\int_{\xi-\tau}^{\xi-\tau_0}L\Big[|\Delta x(s)|+|x_0(s-\tau)-x_0(s-\tau_0)|+|\delta u(s)|$$

$$+|\delta u(s-\theta)|\Big]ds\Big|\le o(\delta w) \tag{4.12}$$

(see (3.1),(4.8)).

According to (3.1), (4.6) for expressions $b_{2,i}(t;\delta w),i=\overline{1,6}$ we obtain

$$|b_{21}(t;\delta w)|\le\|Y\|O(\delta w)\sigma_x(\delta w),\sigma_x(\delta w)=\int_{t_{00}}^{t_1}|\sigma_x(\xi;\delta w)|d\xi;$$

$$|b_{22}(t;\delta w)|\le\|Y\|O(\delta w)\sigma_y(\delta w),\sigma_y(\delta w)=\int_{t_{00}}^{t_1}|\sigma_y(\xi;\delta w)|d\xi;$$

$$|b_{23}(t;\delta w)|\le\|Y\|O(\delta w)\sigma_u(\delta w),\sigma_u(\delta w)=\int_{t_{00}}^{t_1}|\sigma_u(\xi;\delta w)|d\xi;$$

$$|b_{24}(t;\delta w)|\le\|Y\|O(\delta w)\sigma_v(\delta w),\sigma_v(\delta w)=\int_{t_{00}}^{t_1}|\sigma_v(\xi;\delta w)|d\xi;$$

$$b_{25}(t;\delta w)=-\Big[\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau+\gamma_1(t;\delta w),$$

where

$$\|Y\|=\sup\{|Y(\xi;t)|:(\xi,t)\in\Pi\},\gamma_1(t;\delta w)=\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\gamma(\xi;\delta w)d\xi$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$\lim_{\delta w\to 0}\sigma_x(\delta w)=0,\ \lim_{\delta w\to 0}\sigma_y(\delta w)=0,\ \lim_{\delta w\to 0}\sigma_u(\delta w)=0,\ \lim_{\delta w\to 0}\sigma_v(\delta w)=0$$

and

$$\lim_{\delta w\to 0}\frac{|\gamma(t;\delta w)|}{|\delta w|}=0,$$

uniformly for $t\in[t_{00},t_1]$, (see (4.7), (4.9)). Thus,

$$b_{2i}(t;\delta w)=o(\delta w), i=\overline{1,4};$$

$$b_{25}(t;\delta w)=-\Big[\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\dot x_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\delta w).$$

Further,

$$|b_{26}(t;\delta w)|\le\|Y\|\int_{t_{00}}^{t_1}|f_y[\xi]||\Delta x(\xi-\tau)-\Delta x(\xi-\tau_0)|d\xi=o(\delta w)$$

(see (4.10)-(4.12)). Consequently,

$$b_2(t;t_{00},\delta w)=-\Big[\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\dot x_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\delta w)\tag{4.13}$$

From (4.3) by virtue of (4.4), (4.5) and (4.13), we obtain (2.4), where $\delta x(t;\delta w)$ has the form (2.5).

5. Proof of Theorem 2.2

Let $\hat t_0\in(t_{00},t_{00}+\delta)$, where $\delta>0$ and $t_{00}+\delta<t_{02}$. Moreover, let $\varepsilon_2\in(0,\varepsilon_1)$ be insomuch small that $t_0=t_{00}+\delta t_0<\hat t_0$ for arbitrary

$$\delta w\in\delta W^+_{\varepsilon_2}=\Big\{\delta w\in W-w_0:\ |\delta w|\le\varepsilon_2,\ \delta t_0>0\Big\}.$$

The function $\Delta x(t)$ satisfies the equation (4.1) on the interval $[t_0,t_1]$. Therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t)=Y(t_0;t)\Delta x(t_0)+\int_{t_0}^{t}Y(s;t)\Big(f_u[s]\delta u(s)+f_v[s]\delta u(s-\theta)\Big)ds$$
$$+b_1(t;t_0,\delta w)+b_2(t;t_0,\delta w),\tag{5.1}$$

where $Y(\xi;t)$ is the matrix function satisfying the equation (2.8) and the condition (2.9). The function $Y(\xi;t)$ is continuous on the set $[t_{00},\hat t_0)\times[\hat t_0,t_1]$, therefore

$$Y(t_0;t)\Delta x(t_0)=Y(t_{00};t)[\delta\varphi(t_{00})+(\dot\varphi(t_{00})-f_0^+)\delta t_0]+o(t;\delta w)\tag{5.2}$$

(see Theorem 3.3). It is not difficult to see that

$$b_1(t;t_0,\delta w)=\int_{t_0-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi+\int_{t_{00}}^{t_0} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\Delta x(\xi)d\xi$$

$$=\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi+o(t;\delta w)-\int_{t_{00}-\tau_0}^{t_0-\tau_0} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

$$=\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi+o(t;\delta. \quad (5.3)$$

In a similar way, with nonessential changes, for $t\in[\hat{t}_0,t_1]$, one can prove

$$b_2(t;t_0,\delta w)=-\Big[\int_{t_{00}}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big]\delta\tau+o(t;\delta w) \quad (5.4)$$

Taking into account (5.2)-(5.4), from (5.1) we obtain the formula (2.4) on the interval $[\hat{t}_0,t_1]$, where $\delta x(t;\delta w)$ has the form (2.14).

R E F E R E N C E S

1. Gamkrelidze R. Principles of Optimal Control Theory. *Plenum Press, New York*, 1978.

2. Neustadt L.W. Optimization: A Theory of Necessary Conditions. *Princeton Univ. Press, Princeton,* 1976.

3. Ogustoreli N.M. Time-Delay Control Systems. *Academic Press, New York,* 1966.

4. Gabasov R., Kirillova F. The qualitative theory of optimal processes (Russian). *Nauka Moscow*, 1971.

5. Kharatishvili G.L., Tadumadze T.A. Formulas for the variation of a solution and optimal control problems for differential equations with retarded arguments. *J. Math. Sci. (N.Y.)*, **140**, 1 (2007), 1-175.

6. Tadumadze T. Variation formulas of solutions for functional differential equations with several constant delays and their applications in optimal control problems. *Mem. Differ. Equ. Math. Phys.*, **70** (2017), 7-97.

7. Iordanishvili M., Shavadze T., Tadumadze T. Necessary optimality conditions of delay parameters for the nonlinear optimization problem with the mixed initial condition. *Communications in Optimization Theory 2023*, **2** (2023), 1-8.

8. Iordanishvili M., Shavadze T., Tadumadze T. Delay optimization problem for one class of functional differential equation. *Springer Proceedings in Mathematics and Statistics,* **379** (2020), 177-186.

9. Tadumadze T. Sensitivity analysis of one class of controlled functional differential equation considering variable delay perturbation and the continuous initial condition. *International Workshop on the Qualitative Theory of Differential Equations,* December 18-20,2014, Tbilisi, Georgia, Abstracts, 143-146.

10. Tadumadze T., Nachaoui A. On the representations of sensitivity coefficients for nonlinear delay functional differential equations with the discontinuous initial condition.*International Workshop on the Qualitative Theory of Differential Equations,* December 27-29,2015, Tbilisi, Georgia, Abstracts, 157-160.

11. Tadumadze T., Nachaoui A., Shavadze T. On the coefficient of sensitivity of a controlled differential model of the immune response. *Semin. I. Vekua Inst. Appl. Math., Rep.* **48** (2022), 57-64.

12. Tadumadze T., Dvalishvili Ph., Shavadze T. On the representation of solution of the perturbed controlled differential equation with delay and continuous initial condition. *Appl. Comput. Math.* **18**, 3 (2019), 305-315.

13. Nachaoui A., Shavadze T., Tadumadze T. The local representation formula of solution for the perturbed controlled differential equation with delay and discontinuous initial condition. *Mathematics*, **8**, 10 (2020), 18-45.

Received 17.07.2024; revised 15.09.2024; accepted 17.10.2024

Authors' addresses:

K. Berdawood
Salahaddin University-Erbil
Department of Mathematics-College of Science
Kirkuk Road, Erbil
Iraq
Email: karzan.berdawood@su.edu.krd

G. Berikelashvili
A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University
2, Merab Aleksidze II Lane, Tbilisi, 0193
Georgia
E-mail: givi.berikelashvili@ tsu.ge

Ph. Dvalishvili
I. Javakhishvili Tbilisi State University
Department of Computer Sciences
11, University St., Tbilisi, 0186
Georgia
email: pridon.dvalishvili@tsu.ge

T. Tadumadze
I. Javakhishvili Tbilisi State University
Department of Mathematics
I. Vekua Institute of Applied Mathematics
11, University St., Tbilisi, 0186
Georgia
E-mail: tamaz.tadumadze@tsu.ge