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LOCAL REPRESENTATION FORMULAS OF SOLUTION AND OPTIMIZATION PROBLEMS FOR THE CONTROLLED FUNCTIONAL-DIFFERENTIAL EQUATION WITH THE MIXED INITIAL CONDITION

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Abstract. Local representation formulas of solution for functional-differential equations with the mixed initial condition and constant delays in phase coordinates and controls are proved. The necessary conditions of optimality are obtained for the optimization problem with the mixed initial condition.

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1. Introduction

Control systems, whose behavior at a given moment depends on the state of the system in the past are typically described by functional-differential equations with delays [1-6]. In this work, the problems related to the formulas of representation of the solution and optimization problems are discussed for the functional-differential equation

$$\begin{gathered}\dot{x}(t)=(\dot{p}(t),\dot{q}(t))^T=f(t,x(t),p(t-\tau),\\ q(t-\sigma),u(t),u(t-\rho),v(t),v(t-\theta)),\quad x(t)\in R^n\end{gathered}\tag{1.1}$$

with the mixed initial condition

$$x(t)=(p(t),q(t))^T=(\varphi(t),g(t))^T,\ \ t<t_0,\ \ x(t_0)=(p_0,g\,(t_0))^T.\tag{1.2}$$

These problems are closely related to each other. Namely, the necessary conditions of optimality of the element is essentially based on the formulas of representation of the solution.

The condition (1.2) is called a mixed initial condition, because it consists of two parts: the first-discontinuous part, $p(t)=\varphi(t)$, $t<t_0$, $p(t_0)=p_0$, where, in general, $\varphi\,(t_0)\neq p_0$. The discontinuity at the initial moment may be related to an instant change in the dynamic process (change of investment, environmental conditions, concentration of viruses and etc.); Second-continuous part, $q\,(t)=g\,(t)$, $t\leq t_0$, because the condition $q(t_0)=g(t_0)$ is always fulfilled.

(1.1), (1.2) is called the Cauchy problem corresponding to the element

$$\mu=(t_0,\tau,\sigma,\theta,p_0,\varphi(t),g(t),u(t),v(t))$$

The element $\mu_0=(t_{00},\tau_0,\sigma_0,\theta_0,p_{00},\varphi_0(t),g_0(t),u_0(t),v_0(t))$ is called the initial element and problem, corresponding to this element is called the initial problem

$$\dot{x}(t)=f(t,x(t),p(t-\tau_0),q(t-\sigma_0),u_0(t),u_0(t-\rho),v_0(t),v_0(t-\theta_0)),\tag{1.3}$$

$$x(t)=(\varphi_0(t),g_0(t))^T,\quad t<t_{00},\quad x(t_{00})=(p_{00},g\,(t_{00}))^T;\tag{1.4}$$

$\delta\mu = \mu - \mu_0$ is called the variation of the element μ_0. It is clear, that $\mu = \mu_0 + \delta\mu$. So, the element $\mu = \mu_0 + \delta\mu$ is obtained by perturbation of the element μ_0.The problem (1.1), (1.2) is obtained by perturbation of the problem (1.3)–(1.4). Let t_{10} be the endpoint of some interval and assume that the solutions $x(t;\mu_0)$ and $x(t;\mu_0+\delta\mu)$ of the problems (1.3)–(1.4) and (1.1)–(1.2) exist at a neighborhood of the point t_{10}, respectively. The formulas of representation of the solution at a neighborhood of the point t_{10} are proved under the conditions of small variation of $\delta\mu$. Namely, the following representation is proved

$$x(t;\mu_0+\delta\mu) = x(t;\mu_0) + \delta x(t;\delta\mu) + o(t;\delta\mu), \tag{1.5}$$

where $\delta x\,(t;\delta\mu)$ is a linear operator with respect to $\delta\mu$ and $o\,(t;\delta\mu)$ is a high-order infinitesimally small compared to smallness of $\delta\mu$ uniformly with respect to t. The analytical form of the operator $\delta x(t;\delta\mu)$ is established, revealing the effects of the mixed initial condition and the perturbation of the initial data. By initial data we mean the set comprising the initial moment, constant delays included in the phase coordinates, the initial vector, the initial and control functions. The formula of representation of the solution (1.5) plays a crucial role in proving the necessary conditions of optimality [7-14]. Moreover, these formulas are used to find the approximate solution of the perturbed equation. A novel aspect of this paper, unlike the cases considered in [15-17], is that the equation contains two types of control, and a wide class of initial data is considered, including mixed initial condition, for the first time. The formulas of representation of the solution for ordinary differential equations were first proved by R. Gamkrelidze [9], and for various classes of functional differential equations, when the initial moment is fixed, were proved in [7, 8, 12, 13]. T. Tadumadze [18] first obtained and revealed the formulas of representation of the solution and the effects of the discontinuous initial condition, in the case of variation of the initial moment. For different classes of functional-differential equations, the formulas of representation of the solution taking into account the variation of the initial moment, as well as discontinuous and continuous initial conditions were proved in [6, 11, 14, 19, 20]. In the second paragraph, various types of theorems related to the representation of the solution are presented. The type of theorem depends on the property of the right-hand side of the equation, and the combination of variations t_{00} and τ_0 (from the right, from the left, from both sides, etc.). The effects of mixed initial condition and initial data perturbation are also described here.Furthemore, the equation in variations is written out. An analytical form of the approximate solution of the perturbed equation is presented. There are described two possibilities of finding it. The formula of variation of the solution is specified for the linear functional-differential equation. The formulas of representation of the solution are proved, following the scheme given in [6].

In the third paragraph, the problem for the functional-differential equation, with two types of control and mixed initial condition is considered. The optimization problem with general boundary conditions and the functional is also addressed:

$$\begin{cases} \dot{x}(t) = f(t, x(t), p(t-\tau), q(t-\sigma), u(t), u(t-\rho), v(t), v(t-\theta)), \;\; t \in [t_0, t_1], \\ x(t) = (\varphi(t), g(t))^T, \;\; t < t_0, \quad x(t_0) - (p_0, g(t_0))^T, \\ z^i(t_0, t_1, \tau, \sigma, \theta, p_0, g(t_0), x(t_1)) = 0, \;\; i = 1, \dots, l, \\ z^0(t_0, t_1, \tau, \sigma, \theta, p_0, g(t_0), x(t_1)) \to \min. \end{cases}$$

Necessary conditions of optimality of the element $(t_0, t_1, \tau, \sigma, \theta, p_0, \varphi\,(t)\,, g\,(t)\,, u\,(t), \,v\,(t))$ are obtained: in the form of inequalities and equalities-for optimality of t_{00}, t_{10} and τ (the mentioned types depend on the type of variation (from the left, from the right, from both sides)); in the form of equations-for optimality σ_0 and θ_0; in the form of the maximum principle-for optimality of $p_{00}, \varphi_0(t), g_0(t), u_0(t), v_0(t)$. The necessary conditions of optimality are proved according to the scheme given in [6, 11].

The necessary condition of optimality–analogue of Pontryagin's maximum principle for the time optimal problem with a fixed initial moment and fixed endpoints, for a controlled functional-differential equation with a constant delay in phase coordinates, was first proved by G. Kharatishvili in his paper [10]. Numerous papers have since extended this result to optimization problems for different classes of controlled functional-differential equations, among them [4, 6-8, 11-13, 21-32]. In contrast to the works mentioned above, the novelty of this paper lies in its focus on a problem under mixed initial conditions. Here, not only is the optimization of traditional initial data addressed, but also the optimal selection of both types of control and the optimal selection of delay parameters included in the phase coordinates and in one of the controls.

2. Local formulas of representation of solution

Let R^n be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, T is the sign of transpose.

$$|x|^2=\sum_{i=1}^{n}(x^i)^2.$$

Let,

$$s_{11}>s_{10}>s_{01}>s_{00},\ \ \tau_2>\tau_1>0,\ \ \sigma_2>\sigma_1>0,\ \ \rho>0,\ \ \theta_2>\theta_1>0$$

be such numbers,that the following inequality is fulfilled:

$$s_{10}-s_{01}>\max\left\{\tau_2,\sigma_2,\rho,\theta_2\right\}.$$

We mean that

$$P\subset R^m,\quad Q\subset R^k,\quad U\subset R^r,\quad V\subset R^l$$

are open bounded sets, moreover

$$x=(p,q)^T\in O=(P,Q)^T\subset R^n,\quad n=m+k;$$

vector function

$$\begin{gathered}f(t,x,p_1,q_1,u,\omega,v,w)\\=(f^1(t,x,p_1,q_1,u,\omega,v,w),\dots,f^n(t,x,p_1,q_1,u,\omega,v,w))^T\end{gathered}$$

is continuous on the set $I\times O\times P\times Q\times U^2\times V^2$, where $I=[s_{00},s_{11}]$ and is continuously differentiable with respect to the variables x,p_1,q_1,u,ω,v,w. Let Φ and G be the sets of continuously differentiable initial functions.

$$\varphi\left(t\right)\in P,\quad t\in I_1=\left[\widehat{\tau},s_{01}\right],\quad \widehat{\tau}=s_{00}-\max\left\{\tau_2,\sigma_2\right\}$$

and $g(t)\in Q$, $t\in I_1$. Let Ω be a set of piecewise-continuous functions $u(t)\in U$, $t\in I_2=[\widehat{\rho},s_{11}]$, $\widehat{\rho}=s_{00}-\rho$, satisfying the condition: $clu(I_2)$ is compact and $clu\left(I_2\right)\subset U$. Let W be a set of continuously differentiable control functions $v(t)\in V$, $t\in I_3=[\widehat{\theta},s_{10}], \widehat{\theta}=s_{00}-\theta_2$.

To every element

$$\begin{gathered}\mu=(t_0,\tau,\sigma,\theta,p_0,\varphi(t),g(t),u(t),v(t))\in\Lambda\\=(s_{00},s_{01})\times(\tau_1,\tau_2)\times(\sigma_1,\sigma_2)\times(\theta_1,\theta_2)\times P\times\Phi\times G\times\Omega\times W\end{gathered}$$

there corresponds the controlled functional-differential equation:

$$\dot{x}(t) = (\dot{p}(t), \dot{q}(t))^T$$
$$= f\left(t, x\left(t\right), p\left(t-\tau\right), q\left(t-\sigma\right), u\left(t\right), u\left(t-\rho\right), v\left(t\right), v\left(t-\theta\right)\right) \tag{2.1}$$

with the initial condition

$$x(t) = (\varphi(t), g(t))^T, \quad t\in[\widehat{\tau}, t_0), \quad x(t_0) = (p_0, g\left(t_0\right))^T. \tag{2.2}$$

Definition 2.1. Let $\mu\in\Lambda$. A function $x(t) = x(t;\mu)\in O$, $t\in[\widehat{\tau}, t_1]$, $t_1\in(s_{10}, s_{11})$ is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies condition (2.2), and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.1) almost everywhere on $t\in[t_0, t_1]$.

The existence and uniqueness of a corresponding solution locally, on the interval $[\widehat{\tau}, t_0 + \delta]$, is provided by the conditions required for the right-hand side of the equation, where $\delta > 0$ is a small number.

Let's introduce the following notations

$$|\mu| = |t_0| + |\tau| + |\sigma| + |\theta| + |p_0| + ||\varphi||_1 + ||g||_1 + ||u|| + ||v||_1,$$

where,

$$||\varphi||_1 = \sup\left\{|\varphi(t)| + |\dot{\varphi}(t)| : t\in I_1\right\},$$
$$||u|| = \sup\left\{|u(t)| : t\in I_2\right\}, \quad ||v||_1 = \sup\left\{|v(t)| + |\dot{v}(t)| : t\in I_3\right\};$$

$\varepsilon > 0$ is a fixed number and $\mu_0 = \left(t_{00}, \tau_0, \sigma_0, \theta_0, p_{00}, \varphi_0\left(t\right), g_0\left(t\right), u_0\left(t\right),\ v_0\left(t\right)\right)\in\Lambda$ is a fixed element; The variation of t_{00}, τ_0, σ_0, θ_0, p_{00}, $\varphi_0\left(t\right)$, $g_0\left(t\right)$, $u_0\left(t\right)$, $v_0\left(t\right)$ and μ_0 is the following:

$$\delta t_0 = t_0 - t_{00}, \quad \delta\tau = \tau - \tau_0, \quad \delta\sigma = \sigma - \sigma_0, \quad \delta\theta = \theta - \theta_0,$$
$$\delta p_0 = p_0 - p_{00}, \quad \delta\varphi\left(t\right) = \varphi\left(t\right) - \varphi_0\left(t\right), \quad \delta g\left(t\right) = g\left(t\right) - g_0\left(t\right),$$
$$\delta u\left(t\right) = u\left(t\right) - u_0\left(t\right), \quad \delta v(t) = v(t) - v_0(t),$$
$$\delta\mu = \mu - \mu_0.$$

It is clear, that,

$$\begin{cases} t_0 = t_{00} + \delta t_0,\ \tau = \tau_0 + \delta\tau,\ \sigma = \sigma_0 + \delta\sigma,\ \theta = \theta_0 + \delta\theta, \\ \qquad p_0 = p_{00} + \delta p_0, \\ \varphi(t) = \varphi_0(t) + \delta\varphi(t),\ g(t) = g_0(t) + \delta g(t),\ u(t) = u_0(t) + \delta u(t), \\ \qquad v(t) = v_0(t) + \delta v(t), \\ \mu = \mu_0 + \delta\mu; \end{cases} \tag{2.3}$$

Further,

$$\Lambda_\varepsilon\left(\mu_0\right) = \left\{\mu\in\Lambda : |\mu - \mu_0| \leq\varepsilon\right\}, \quad \Lambda_\varepsilon(\mu_0) - \mu_0 = \left\{\mu - \mu_0 : \mu\in\Lambda_\varepsilon(\mu_0)\right\};$$

We can write $\Lambda_\varepsilon\left(\mu_0\right) - \mu_0$ in the form:

$$\Lambda_\varepsilon(\mu_0) - \mu_0 = \left\{\delta\mu\in\Lambda - \mu_0 : |\delta\mu|\leq\varepsilon\right\}.$$

Let $x_0(t)$ be the solution corresponding to the element μ_0, defined on the interval $[\widehat{\tau}, t_{10}]$, $t_{10}\in\left(s_{10}, s_{11}\right)$. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$, such that to an arbitrary element

$\mu\in\Lambda_{\varepsilon_1}(\mu_0)$ there corresponds the solution $x\left(t;\mu\right)$, defined on the interval $[\widehat{\tau},t_{10}+\delta_1]$, and $(t_{10}-\delta_1,t_{10}+\delta_1)\in(s_{10},s_{11})$. Due to the uniqueness, the solution $x(t;\mu_0)$ is the continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Therefore, the solution $x_0\left(t\right)$, in the sequel, is assumed to be defined on the interval $[\widehat{\tau},t_{10}+\delta_1]$.

Let us introduce the increment of the solution $x_0(t)=x(t;\mu_0)$

$$x\left(t;\mu_0+\delta\mu\right)-x_0\left(t\right)=\Delta x\left(t;\delta\mu\right),\quad \delta\mu\in\Lambda_{\varepsilon_1}\left(\mu_0\right)-\mu_0, t\in[\widehat{\tau},\ \ t_{10}+\delta_1].$$

Theorem 2.1. *Let $x_0(t)$ be the solution, corresponding to the element μ_0, defined on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Then there exists a number $\varepsilon_2\in(0,\varepsilon_1)$, such, that for an arbitrary $t\in[t_{10}-\delta_1,t_{10}+\delta_1]$ $\delta\mu\in\Lambda^-_{\varepsilon_2}(\mu_0)-\mu_0$*

$$x\left(t;\mu_0+\delta\mu\right)=x_0\left(t\right)+\delta x^-\left(t;\delta\mu\right)+o\left(t;\delta\mu\right), \tag{2.4}$$

where,

$$\begin{gathered}\Lambda^-_{\varepsilon_2}(\mu_0)-\mu_0=\left\{\delta\mu\in\Lambda_{\varepsilon_2}(\mu_0)-\mu_0:\delta t_0\leq 0,\ \delta\tau\leq 0\right\},\\ \delta x^-(t;\delta\mu)=Y(t_{00};t)(\delta p_0,\delta g(t_{00}))^T\\ +\left\{Y(t_{00};t)\left[\left(\theta_{m\times1},\dot g_0(t_{00})\right)^T-f_0^-\right]-Y\left(t_{00}+\tau_0;t\right)f_{01}^-\right\}\delta t_0\\ -Y(t_{00}+\tau_0;t)f_{01}^-\delta\tau+\gamma\left(t;\delta\mu\right),\end{gathered} \tag{2.5}$$

$$\begin{gathered}\gamma(t;\delta\mu)=-\left\{\int_{t_{00}}^{t}Y(\xi;t)f_{p_1}[\xi]\dot p_0(\xi-\tau_0)d\xi\right\}\delta\tau\\ -\left\{\int_{t_{00}}^{t}Y(\xi;t)f_{q_1}[\xi]\dot q_0(\xi-\sigma_0)d\xi\right\}\delta\sigma\\ -\left\{\int_{t_{00}}^{t}Y(\xi;t)f_w[\xi]\dot v_0(\xi-\theta_0)d\xi\right\}\delta\theta+\int_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{p_1}[\xi+\tau_0]\delta\varphi(\xi)d\xi\\ +\int_{t_{00}-\sigma_0}^{t_{00}}Y(\xi+\sigma_0;t)f_{q_1}[\xi+\sigma_0]\delta g(\xi)d\xi+\int_{t_{00}}^{t}Y(\xi;t)(f_u[\xi]\delta u(\xi)\\ +f_\omega[\xi]\delta u(\xi-\rho))d\xi+\int_{t_{00}}^{t}Y(\xi;t)(f_v[\xi]\delta v(\xi)+f_w[\xi]\delta v(\xi-\theta_0))d\xi;\end{gathered} \tag{2.6}$$

$$\lim_{\delta\mu\to0}\frac{o(t;\delta\mu)}{|\delta\mu|}=0.$$

uniformly with respect to $t\in[t_{10}-\delta_1,t_{10}+\delta_1]$;

$$\begin{gathered}f_0^-=f(t_{00},x_0(t_{00}),\varphi_0(t_{00}-\tau_0),g_0(t_{00}-\sigma_0),u_0(t_{00}-),u_0(t_{00}-\rho-),\\ v_0(t_{00}),v_0(t_{00}-\theta_0)),\\ f_{01}^-=f(t_{00}+\tau_0,x_0(t_{00}+\tau_0),p_{00},q_0(t_{00}+\tau_0-\sigma_0),u_0(t_{00}+\tau_0-),\\ u_0(t_{00}+\tau_0-\rho-),\\ v_0(t_{00}+\tau_0),v_0(t_{00}+\tau_{00}-\theta_0)-f(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}),\\ q_0(t_{00}+\tau_0-\sigma_0),u_0(t_{00}+\tau_0-,u_0(t_{00}+\tau_0-\rho-,v_0(t_{00}+\tau_0),\\ v_0(t_{00}+\tau_{00}-\theta_0)),\\ f_p[\xi]=f_p(\xi,x_0(\xi),p_0(\xi-\tau_0),q_0(\xi-\sigma_0),u_0(\xi),u_0(\xi-\rho),v_0(\xi),v_0(\xi-\theta_0)).\end{gathered}$$

Moreover $Y\,(\xi;t)$ is $n{\times}n$ dimensional matrix function, satisfying the functional-differential matrix equation with the advanced argument on the interval (t_{00},t)

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]$$
$$-(Y(\xi+\tau_0;t)f_{p_1}[\xi+\tau_0]\;Y(\xi+\sigma_0;t)f_{q_1}[\xi+\sigma_0]),\quad \xi{\in}(t_{00},t),t\in(t_{10}-\delta_1,t_{10}+\delta_1) \quad (2.7)$$

and the initial condition

$$Y(\xi;t)=\begin{cases}E_{n\times n}, & \xi=t,\\ \Theta_{n\times n}, & \xi>t,\end{cases} \quad (2.8)$$

on the interval $[t,\infty)$. $E_{n\times n}$ is the identity matrix and $\Theta_{n\times n}$ is the zero matrix.

Some comments. $\delta x^-(t;\delta\mu)$ is called the first variation of the solution $x_0(t)$ on the interval $t{\in}\,[t_{10}-\delta_1,t_{10}+\delta_1]$. The linear operator (2.5), where $\gamma\,(t;\delta\mu)$ has the form (2.6) is called the local formula of variation of the solution. In the paper, an essential novelty is the fact that,when proving the formulas of the variation of the solution, in addition to previously known variations, in conditions of the mixed initial condition, two types of control and the variation of the delay parameter included in the control $v_0(t)$ are considered.

The expression in the formula (2.5)

$$Y(t_{00};t)(\delta p_0,\delta g(t_{00}))^T$$
$$+\{Y(t_{00};t)[(\Theta_{m\times1},\dot g_0(t_{00}))^T-f_0^-]-Y(t_{00}+\tau_0;t)f_{01}^-\}\delta t_0$$

is the effect of the mixed initial condition.

The expression

$$\{Y(t_{00};t)[(\Theta_{m\times1},\dot g_0(t_{00}))^T-f_0^-]-Y(t_{00}+\tau_0;t)f_{01}^-\}\delta t_0$$

is the effect of the left variation of the initial moment t_{00}.

The expression

$$Y(t_{00};t)(\delta p_0,\delta g(t_{00}))^T$$

is the effect of the variation of the initial vector p_{00} and of the initial function $g_0(t)$.

f_{01}^- is the effect of discontinuity at the initial moment $p_0\,(t)$.

$(\Theta_{m\times1},\dot g_0(t_{00}))^T$ is the effect of continuity at the initial moment $q_0(t)$.

In the formula (2.6):

$$\left\{\int_{t_{00}}^{t}Y(\xi;t)f_{p_1}[\xi]\dot p_0(\xi-\tau_0)\,d\xi\right\}\delta\tau$$

is the effect of the variation of the delay parameter τ_0;

$$\left\{\int_{t_{00}}^{t}Y(\xi;t)f_{q_1}[\xi]\dot q_0(\xi-\sigma_0)\,d\xi\right\}\delta\sigma$$

is the effect of the variation of the delay parameter σ_0;

$$\left\{\int_{t_{00}}^{t}Y(t_{00};t)f_w[\xi]\dot v_0(\xi-\theta_0)\,d\xi\right\}\delta\theta$$

is the effect of the variation of the delay parameter θ_0;

$$\int_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{p_1}[\xi+\tau_0]\delta\varphi(\xi)\,d\xi+\int_{t_{00}-\sigma_0}^{t_{00}}Y(\xi+\sigma_0;t)f_{q_1}[\xi+\sigma_0]\delta g(\xi)\,d\xi$$

is the effect of the variation of initial functions;

$$\int_{t_{00}}^{t} Y(\xi;t)(f_u[\xi]\delta u(\xi)+f_\omega[\xi]\delta u(\xi-\rho))d\xi$$

is the effect of the variation of control $u_0(t)$;

$$\int_{t_{00}}^{t} Y(\xi;t)[f_v[\xi]\delta v(\xi)+f_w[\xi]\delta v(\xi-\theta_0)]d\xi$$

is the effect of the variation of control $v_0\,(t)$.

It is clear, that,

$$\delta x^-(t;\delta\mu)=\delta x_1^-(t;\delta\mu)+\delta x_2^-(t;\delta\mu),$$

where,

$$\begin{aligned}\delta x_1^-(t;\delta\mu)&=Y(t_{00};t)\{(\delta p_0,\delta g(t_{00}))^T+[(\Theta_{m\times 1},\dot g_0(t_{00}))^T\\&-f_0^-]\delta t_0\}-\left\{\int_{t_{00}}^{t} Y(\xi;t)f_{p_1}[\xi]\dot p_0(\xi-\tau_0)d\xi\right\}\delta\tau\\&-\left\{\int_{t_{00}}^{t} Y(\xi;t)f_{q_1}[\xi]\dot q_0(\xi-\sigma_0)d\xi\right\}\delta\sigma\\&-\left\{\int_{t_{00}}^{t} Y(t_{00};t)f_w[\xi]\dot v_0(\xi-\theta_0)d\xi\right\}\delta\theta\\&+\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_{p_1}[\xi+\tau_0]\delta\varphi(\xi)d\xi\\&+\int_{t_{00}-\sigma_0}^{t_{00}} Y(\xi+\sigma_0;t)f_{q_1}[\xi+\sigma_0]\delta g(\xi)d\xi\\&+\int_{t_{00}}^{t} Y(\xi;t)[f_v[\xi]\delta v(\xi)+f_w[\xi]\delta v(\xi-\theta_0)]d\xi;\\\delta x_2^-(t;\delta\mu)&=-Y(t_{00}+\tau_0;t)f_{01}^-(\delta t_0+\delta\tau).\end{aligned}$$

Based on the Cauchy formula, we conclude, that the function

$$\delta x_1^-(t)=\begin{cases}(\delta\varphi(t),\delta g(t))^T, & t\in[\hat\tau,t_{00}),\\(\delta p_0,\delta g(t_{00}))^T, & t=t_{00},\\\delta x_1^-(t;\delta\mu), & t\in(t_{00},t_{10}+\delta_1]\end{cases}$$

is the solution of the following differential equation:

$$\begin{aligned}\dot{\delta x}(t)&=f_x[t]\,\delta x(t)+f_{p_1}[t]\,\delta p(t-\tau_0)+f_{q_1}[t]\,\delta q(t-\sigma_0)\\&-f_{p_1}[t]\dot p_0(t-\tau_0)\delta\tau-f_{q_1}[t]\dot q_0(t-\sigma_0)\delta\sigma\\&-f_w[t]\dot v_0(t-\theta_0)\delta\theta+f_u[t]\delta u(t)+f_\omega[t]\delta u(t-\rho)+f_v[t]\delta v(t)\\&+f_w[t]\delta v(t-\theta_0),\quad t\in[t_{00},t_{10}+\delta_1]\end{aligned}\tag{2.9}$$

with the initial condition

$$\delta x(t)=(\delta\varphi(t),\delta g(t))^T,t\in[\tau,t_{00}),\quad \delta x(t_{00})=(\delta p_0,\delta g\,(t_{00}))^T\tag{2.10}$$

(2.9) is called the equation in "variations".

The function

$$\delta x_2^-(t) = \begin{cases} 0, & t\in[\widehat{\tau}, t_{00}+\tau_0), \\ -f_{01}^-[\delta t_0+\delta\tau], & t=t_{00}+\tau_0, \\ \delta x_2^-(t;\delta\mu), & t\in(t_{00}+\tau_0, t_{10}+\delta_1] \end{cases}$$

is the solution of the following differential equation:

$$\dot{\delta x}(t) = f_x[t]\delta x(t) + f_{p_1}[t]\delta p(t-\tau_0) + f_{q_1}[t]\delta q(t-\sigma_0), \quad t\in\left[t_{00}+\tau_0, t_{10}+\delta_1\right]. \tag{2.11}$$

With the initial condition

$$\delta x\left(t\right)=0, t\in\left[\widehat{\tau}, t_{00}+\tau_0\right), \quad \delta x\left(t_{00}+\tau_0\right) = -f_{01}^-\left[\delta t_0+\delta\tau\right]. \tag{2.12}$$

Formula (2.4) allows us to obtain an approximate solution of the perturbed Cauchy problem in the analytical form on the interval $[t_{10}-\delta_1, t_{10}+\delta_1]$,

$$\begin{gathered} \dot{x}(t) = f(t, x(t), p(t-\tau_0-\delta\tau), q(t-\sigma_0-\delta\sigma), u_0(t) \\ +\delta u(t), u_0(t-\rho)+\delta u(t-\rho), \\ v_0(t)+\delta v(t), v_0(t-\theta_0-\delta\theta)+\delta v(t-\theta_0-\delta\theta)), \\ x(t) = (\varphi_0(t)+\delta\varphi(t), g_0(t)+\delta g(t))^T, \quad t\in[\widehat{\tau}, t_{00}+\delta t_0), \\ x(t_{00}+\delta t_0) = (p_{00}+\delta p_0, g_0(t_{00}+\delta t_0)+\delta g(t_{00}+\delta t_0))^T \end{gathered}$$

corresponding to the following element:

$$\begin{gathered} \mu=\mu_0+\delta\mu=(t_{00}+\delta t_0, \tau_0+\delta\tau, \sigma_0+\delta\sigma, \theta_0+\delta\theta, p_{00}+\delta p_0, \varphi_0\left(t\right) \\ +\delta\varphi\left(t\right), g_0\left(t\right)+\delta g\left(t\right), u_0(t)+\delta u(t), v_0(t)+\delta v(t)) \end{gathered}$$

In fact, for sufficiently small $|\delta\mu|$ from (2.4) it follows, that

$$x\left(t;\mu_0+\delta\mu\right) \approx x_0\left(t\right)+\delta x^-\left(t;\delta\mu\right), \quad t\in\left[t_{10}-\delta_1, t_{10}+\delta_1\right]$$

The first variation of solution can be calculated in two ways: one-by finding the matrix function $Y(\xi;t)$ (see (2.7)–(2.8)), the other-by finding the solutions of the equation (2.9)–(2.10) and (2.11)–(2.12).

Let us consider the linear case. Let,

$$\begin{gathered} f(t,x,p_1,q_1,u,\omega,v,w) \\ =A_1(t)x+B_1(t)p_1+C_1(t)q_1+D_1(t)u+D_2(t)\omega+E_1(t)v+E_2(t)w+f_1(t), \end{gathered}$$

Where coefficients and free member are continuous on I.

In this case, for Theorem 2.1 we have

$$\begin{gathered} f_0^- = A_1(t_{00})x_0(t_{00})+B_1(t_{00})p_0(t_{00}-\tau_0)+C_1(t_{00})q_0(t_{00}-\sigma_0) \\ +D_1(t_{00})u_0(t_{00}-)+D_2(t_{00})u_0(t_{00}-\rho-) \\ +E_1(t_{00})v_0(t_{00})+E_2(t_{00})v_0(t_{00}-\theta_0)+f_1(t_{00}), \\ f_{01}^- = B_1(t_{00}+\tau_0)(p_{00}-\varphi_0(t_{00})); \end{gathered}$$

$$\gamma(t;\delta\mu)=-\left\{\int_{t_{00}}^{t}Y(\xi;t)B_1(\xi)\dot{p}_0(\xi-\tau_0)d\xi\right\}\delta\tau$$

$$-\left\{\int_{t_{00}}^{t}Y(\xi;t)C_1(\xi)\dot{q}_0(\xi-\sigma_0)d\xi\right\}\delta\sigma-\left\{\int_{t_{00}}^{t}Y(\xi;t)E_0(\xi)\dot{v}_0(\xi-\theta_0)d\xi\right\}\delta\theta$$

$$+\int_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)B_1(\xi+\tau_0)\delta\varphi(\xi)d\xi$$

$$+\int_{t_{00}-\sigma_0}^{t_{00}}Y(\xi+\sigma_0;t)C_1(\xi+\sigma_0)\delta g(\xi)d\xi+\int_{t_{00}}^{t}Y(\xi;t)(D_1(\xi)\delta u(\xi)$$

$$+D_2(\xi)\delta u(\xi-\rho))d\xi+\int_{t_{00}}^{t}Y(\xi;t)(E_1(\xi)\delta v(\xi)+E_2(\xi)\delta v(\xi-\theta_0))d\xi;$$

$$Y_\xi(\xi;t)=-Y(\xi;t)A_1(\xi)$$

$$-(Y(\xi+\tau_0;t)B_1(\xi+\tau_0)Y(\xi+\sigma_0;t)C_1(\xi+\sigma_0)),\xi\in(t_{00},t).$$

Theorem 2.2. *Let $x_0(t)$ be the solution, corresponding to the element μ_0, defined on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Then there exists a number $\varepsilon_2\in(0,\varepsilon_1)$, such, that for an arbitrary $t\in[t_{10}-\delta_1,t_{10}+\delta_1]$ and $\delta\mu\in\Lambda^+_{\varepsilon_2}(\mu_0)-\mu_0$*

$$x(t;\mu_0+\delta\mu)=x_0(t)+\delta x^+(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\Lambda^+_{\varepsilon_2}(\mu_0)-\mu_0=\{\delta\mu\in\Lambda_{\varepsilon_2}(\mu_0)-\mu_0:\delta t_0\geq 0,\delta\tau\geq 0\},\ \delta x^+(t;\delta\mu)$$

$$=Y(t_{00};t)(\delta p_0,\delta g(t_{00}))^T$$

$$+\{Y(t_{00};t)[(\Theta_{m\times 1},\dot{g}_0(t_{00}))^T-f_0^+]-Y(t_{00}+\tau_0;t)f_{01}^+\}\delta t_0$$

$$-Y(t_{00}+\tau_0;t)f_{01}^+\delta\tau+\gamma(t;\delta\mu).$$

Theorem 2.3. *Let $x_0(t)$ be the solution, corresponding to the element μ_0, defined on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Let, besides, the functions $u_0(t)$ and $u_0(t-\rho)$ be continuous at the point $t_{00}+\tau_0$. Then there exists a number $\varepsilon_2\in(0,\varepsilon_1)$, such, that for an arbitrary $t\in[t_{10}-\delta_1,t_{10}+\delta_1]$ and $\delta\mu\in\Lambda^-_{\varepsilon_2}(\mu_0)-\mu_0$,*

$$x(t;\mu_0+\delta\mu)=x_0(t)+\delta x^-(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\Lambda^-_{\varepsilon_2}(\mu_0)-\mu_0=\{\delta\mu\in\Lambda_{\varepsilon_2}(\mu_0)-\mu_0:\delta t_0\leq 0\},\delta\widehat{x}^-(t;\delta\mu)$$

$$=Y(t_{00};t)(\delta p_0,\delta g(t_{00}))^T$$

$$+\{Y(t_{00};t)[(\Theta_{m\times 1},\dot{g}_0(t_{00}))^T-f_0^-]-Y(t_{00}+\tau_0;t)f_{01}\}\delta t_0$$

$$-Y(t_{00}+\tau_0;t)f_{01}\delta\tau+\gamma(t;\delta\mu),$$

$$f_{01}^-=f(t_{00}+\tau_0,x_0(t_{00}+\tau_0),p_{00},q_0(t_{00}+\tau_0-\sigma_0),u_0(t_{00}+\tau_0),$$

$$u_0(t_{00}+\tau_0-\rho),v_0(t_{00}+\tau_0),v_0(t_{00}+\tau_{00}-\theta_0))$$

$$-f(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}),q_0(t_{00}+\tau_0-\sigma_0),$$

$$u_0(t_{00}+\tau_0),u_0(t_{00}+\tau_0-\rho),v_0(t_{00}+\tau_0),v_0(t_{00}+\tau_{00}-\theta_0)).$$

Theorem 2.4. *Let $x_0(t)$ be the solution, corresponding to the element μ_0, defined on the interval $[\widehat{\tau}, t_{10}+\delta_1]$. Let, besides, the functions $u_0(t)$ and $u_0(t-\rho)$ be continuous at the point $t_{00}+\tau_0$. Then there exists a number $\varepsilon_2 \in (0,\varepsilon_1)$, such, that for an arbitrary $t \in [t_{10}-\delta_1, t_{10}+\delta_1]$ and $\delta\mu \in \Lambda^{+}_{\varepsilon_2}(\mu_0)-\mu_0$,*

$$x(t;\mu_0+\delta\mu) = x_0(t) + \delta x^{+}(t;\delta\mu) + o(t;\delta\mu),$$

where

$$\widehat{\Lambda}^{+}_{\varepsilon_2}(\mu_0)-\mu_0 = \{\delta\mu \in \Lambda_{\varepsilon_2}(\mu_0)-\mu_0 : \delta t_0 \geq 0\},$$
$$\delta x^{+}(t;\delta\mu) = Y(t_{00};t)(\delta p_0, \delta g(t_{00}))^T + \{Y(t_{00};t)[(\Theta_{m\times 1}, \dot{g}_0(t_{00}))^T - f_0^{+}]$$
$$-Y(t_{00}+\tau_0;t)f_{01}\}\delta t_0 - Y(t_{00}+\tau_0;t)f_{01}\delta\tau + \gamma(t;\delta\mu).$$

Theorem 2.5. *Let $x_0(t)$ be the solution, corresponding to the element μ_0, defined on the interval $[\widehat{\tau}, t_{10}+\delta_1]$. Let, besides, the functions $u_0(t)$ and $u_0(t-\rho)$ be continuous at the points t_{00} and $t_{00}+\tau_0$. Then there exists a number $\varepsilon_2 \in (0,\varepsilon_1)$, such, that for an arbitrary $t \in [t_{10}-\delta_1, t_{10}+\delta_1]$ and $\delta\mu \in \Lambda_{\varepsilon_2}(\mu_0)-\mu_0$,*

$$x(t;\mu_0+\delta\mu) = x_0(t) + \delta x(t;\delta\mu) + o(t;\delta\mu),$$

where,

$$\delta x(t;\delta\mu) = Y(t_{00};t)(\delta p_0, \delta g(t_{00}))^T$$
$$+\{Y(t_{00};t)[(\Theta_{m\times 1}, \dot{g}_0(t_{00}))^T - f_0] - Y(t_{00}+\tau_0;t)f_{01}\}\delta t_0$$
$$-Y(t_{00}+\tau_0;t)f_{01}\delta\tau + \gamma(\tau;\delta\mu).$$

3. Statement of the problem of optimization and necessary conditions of optimality

Let $P_0 \subset P$, $Q_0 \subset Q$, $U_0 \subset U$, $V_0 \subset V$ be convex sets.

Let us introduce the sets:

$$\Phi_0 = \{\varphi(t) \in P_0 : \varphi(t) \in \Phi\}, \quad G_0 = \{g(t) \in Q_0 : g(t) \in G\},$$
$$\Omega_0 = \{u(t) \in U_0 : u(t) \in \Omega\}, \quad W_0 = \{v(t) \in V_0 : v(t) \in W\}.$$

To each element

$$\vartheta = (t_0, t_1, \tau, \sigma, \theta, p_0, \varphi(t), g(t), u(t), v(t)) \in \Delta$$
$$= (s_{00}, s_{01}) \times (s_{10}, s_{11}) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times (\theta_1, \theta_2)$$
$$\times P_0 \times \Phi_0 \times G_0 \times \Omega_0 \times W_0,$$

we assign the controlled functional-differential equation:

$$\dot{x}(t) = (\dot{p}(t), \dot{q}(t))^T$$
$$= f(t, x(t), p(t-\tau), q(t-\sigma), u(t), u(t-\rho), v(t), v(t-\theta)), \quad t \in [t_0, t_1] \qquad (3.1)$$

With mixed initial condition

$$x(t) = (\varphi(t), g(t))^T, \quad t\in[\widehat{\tau}, t_0), x(t_0) = (p_0, g(t_0))^T \tag{3.2}$$

Definition 3.1. Let $\vartheta\in\Delta$. A function $x(t) = x(t;\vartheta)\in O$, $t\in[\widehat{\tau}, t_1]$ is called a solution of equation (3.1) with the initial condition (3.2) or a solution corresponding to the element ϑ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies condition (3.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (3.1) almost everywhere on $[t_0, t_1]$.

Let, scalar functions $z^i(t_0, t_1, \tau, \sigma, \theta, p, q, x)$, $i = 0, 1, \ldots, l$ be continuously differentiable on the set $[s_{00}, s_{01}] \times [s_{10}, s_{11}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times [\theta_1, \theta_2] \times P \times Q \times O$.

Definition 3.2. The element $\vartheta = (t_0, t_1, \tau, \sigma, \theta, p_0, \varphi(t), g(t), u(t), v(t)) \in\Delta$ is called admissible if the corresponding solution $x(t) = x(t;\vartheta)$ satisfies the boundary conditions

$$z^i(t_0, t_1, \tau, \sigma, \theta, p_0, g(t_0), x(t_1)) = 0, \quad i = 1, \ldots, l. \tag{3.3}$$

The set of admissible elements will be denoted by Δ_0.

Definition 3.3. The element $\vartheta_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, \theta_0, p_{00}, \varphi_0(t), g_0(t), u_0(t), v_0(t)) \in\Delta_0$ is called optimal, if for arbitrary element $\vartheta\in\Delta_0$ the inequality

$$\begin{aligned} &z^0(t_{00}, t_{10}, \tau_0, \sigma_0, \theta_0, p_{00}, g_0(t_{00}), x_0(t_{10})) \\ &\quad\le z^0(t_0, t_1, \tau, \sigma, \theta, p_0, g(t_0), x(t_1)), \end{aligned} \tag{3.4}$$

is fulfilled, where, $x_0(t) = x(t;\vartheta_0)$. (3.1)–(3.4), is called optimization problem for equation (3.1) with mixed initial condition (3.2).

Theorem 3.1. *Let ϑ_0 be an optimal element, and let $x_0(t) = (p_0(t), q_0(t))^T$, $t\in[\widehat{\tau}, t_{10}]$ be the corresponding solution. Then there exists a vector $\pi = (\pi_0, \pi_1, \ldots, \pi_l), \pi_0\le 0$ and a solution $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ of the equation*

$$\begin{aligned} \dot{\psi}(t) = &-\psi(t) f_x[t] - \psi(t+\tau_0)(f_p[t+\tau_0]\Theta_{n\times k}) \\ &-\psi(t+\sigma)(\Theta_{n\times m} f_q[t+\tau_0]), \quad t\in(t_{00}, t_{10}) \end{aligned} \tag{3.5}$$

with the initial condition

$$\psi(t_{10}) = \pi Z_{0x}, \quad \psi(t) = 0, \quad t > t_{10}, \tag{3.6}$$

where

$$Z = (z^0, \ldots, z^l)^T, \quad Z_{0x} = Z_x(t_{00}, t_{10}, \tau_0, \sigma_0, \theta_0, p_{00}, g_0(t_{00}), x_0(t_{10})),$$
$$f_x[t] = f_x(t, x(t), p(t-\tau), q(t-\sigma), u(t), u(t-\rho), v(t), v(t-\theta)).$$

such, that the following conditions are fulfilled:

3.1) The condition for the initial moment t_{00}

$$\pi Z_{0t_0} + [\pi Z_{0q} + (\psi_{k+1}(t_{10}), \ldots, \psi_n(t_{10}))]\dot{g}_0(t_{00}) \ge \psi(t_{00}) f_0^- + \psi(t_{00}+\tau_0) f_{01}^-;$$

3.2) The condition for the final moment t_{10}

$$\pi Z_{0t_1} \le -\psi(t_{10}) f_{02}^-;$$

3.3) The condition for τ_0

$$\pi Z_{0\tau} \leq \psi(t_{00}+\tau_0) f_{01}^{-} + \int_{t_{00}}^{t_{10}} \psi(t) f_p[t] \dot{p}_0(t-\tau_0)dt;$$

3.4) The condition for σ_0

$$\pi Z_{0\sigma} = \int_{t_{00}}^{t_{10}} \psi(t) f_q[t] \dot{q}_0(t-\sigma_0)dt;$$

3.5) The condition for θ_0

$$\pi Z_{0\theta} = \int_{t_{00}}^{t_{10}} \psi(t) f_w[t] \dot{v}_0(t-\theta_0)dt;$$

3.6) The condition for p_{00}

$$\{\pi Z_{0p_0} + (\psi_1(t),\ldots,\psi_k(t))\} p_{00} = \max_{p_0\in P_0} \{\pi Z_{0p_0} + (\psi_1(t),\ldots,\psi_m(t))\} p_0;$$

3.7) The condition for $\varphi_0(t)$

$$\int_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_p[t+\tau_0] \varphi_0(t)dt = \mathop{max}_{\varphi(t)\in\Phi_0} \int_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_p[t+\tau_0] \varphi(t)dt;$$

3.8) The condition for $g_0(t)$

$$\pi Z_{0q} g_0(t_{00}) + \int_{t_{00}-\sigma_0}^{t_{00}} \psi(t+\sigma_0) f_q[t+\sigma_0] g_0(t)dt$$
$$= \max_{g(t)\in G_0} \{\pi Z_{0q} g(t_{00}) + \int_{t_{00}-\sigma_0}^{t_{00}} \psi(t+\sigma_0) f_q[t+\sigma_0] g(t)dt\};$$

3.9) The condition for $u_0(t)$

$$\int_{t_{00}}^{t_{10}} \psi(t)\{f_u[t]u_0(t) + f_\omega[t]u_0(t-\rho)\}dt$$
$$= \max_{u(t)\in\Omega_0} \left[\int_{t_{00}}^{t_{10}} \psi(t)\{f_u[t]u(t) + f_\omega[t]u(t-\rho)\}dt\right];$$

3.10) The condition for $v_0(t)$

$$\int_{t_{00}}^{t_{10}} \psi(t)\{f_v[t]v_0(t) + f_w[t]v_0(t-\theta_0)\}dt$$
$$= \max_{v(t)\in G_0} \left[\int_{t_{00}}^{t_{10}} \psi(t)\{f_v[t]v(t) + f_w[t]v(t-\theta_0)\}dt\right].$$

Theorem 3.1. corresponds to the case, when the left variation is held at points t_{00}, τ_0 and t_{10}. In this case we called it $(---)$ variation.

Theorem 3.2. *Let ϑ_0 be an optimal element, and $x_0(t) = (p_0(t), q_0(t))^T$, $t\in[\widehat{\tau}, t_{10}]$ is the corresponding solution. Then, there exists a vector $\pi = (\pi_0, \pi_1, \ldots, \pi_l), \pi_0 \leq 0$ and a solution*

$\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ *of equation* (3.5) *with initial condition* (3.6), *such that conditions* 3.4) $-$ 3.10) *are fulfilled. Moreover,*

$$\pi Z_{0t_0} + [\pi Z_{0q} + (\psi_{k+1}(t_{10}), \ldots, \psi_n(t_{10}))]\dot{g}_0(t_{00}) \le \psi(t_{00}) f_0^+ + \psi(t_{00} + \tau_0) f_{01}^+;$$
$$\pi Z_{0t_1} \ge -\psi(t_{10}) f_{02}^+;$$
$$\pi Z_{0\tau} \ge \psi(t_{00} + \tau_0) f_{01}^+ + \int_{t_{00}}^{t_{10}} \psi(t) f_p[t] \dot{p}_0(t - \tau_0) dt.$$

Theorem 3.2. corresponds to the case, when we have the variation $(+++)$, i. e., the rightvariation of the points t_{00}, τ_0 and t_{10}.

Theorem 3.3. *Let* ϑ_0 *be an optimal element, and* $x_0(t) = (p_0(t), q_0(t))^T$, $t \in [\widehat{\tau}, t_{10}]$ *is the corresponding solution. Then, there exists a vector* $\pi = (\pi_0, \pi_1, \ldots, \pi_l), \pi_0 \le 0$ *and a solution* $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ *of equation* (3.5) *with initial condition* (3.6), *such that conditions* 3.4) $-$ 3.10) *are fulfilled. Moreover,*

$$\pi Z_{0t_0} + [\pi Z_{0q} + (\psi_{k+1}(t_{10}), \ldots, \psi_n(t_{10}))]\dot{g}_0(t_{00}) \le \psi(t_{00}) f_0^- + \psi(t_{00} + \tau_0) f_{01}^-;$$
$$\pi Z_{0t_1} \ge -\psi(t_{10}) f_{02}^+;$$
$$\pi Z_{0\tau} \ge \psi(t_{00} + \tau_0) f_{01}^+ + \int_{t_{00}}^{t_{10}} \psi(t) f_p[t] \dot{p}_0(t - \tau_0) dt.$$

Theorem 3.3. corresponds to the case, when we have the variation $(-++)$.

Theorem 3.4. *Let* ϑ_0 *be an optimal element, and* $x_0(t) = (p_0(t), q_0(t))^T$, $t \in [\widehat{\tau}, t_{10}]$ *is the corresponding solution. Then, there exists a vector* $\pi = (\pi_0, \pi_1, \ldots, \pi_l), \pi_0 \le 0$ *and a solution* $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ *of equation* (3.5) *with the initial condition* (3.6), *such that conditions* 3.4) $-$ 3.10) *are fulfilled. Moreover,*

$$\pi Z_{0t_0} + [\pi Z_{0q} + (\psi_{k+1}(t_{10}), \ldots, \psi_n(t_{10}))]\dot{g}_0(t_{00}) \ge \psi(t_{00}) f_0^- + \psi(t_{00} + \tau_0) f_{01}^-;$$
$$\pi Z_{0t_1} \le -\psi(t_{10}) f_{02}^-;$$
$$\pi Z_{0\tau} \ge \psi(t_{00} + \tau_0) f_{01}^+ + \int_{t_{00}}^{t_{10}} \psi(t) f_p[t] \dot{p}_0(t - \tau_0) dt.$$

Theorem 3.4. corresponds to the case, when we have the variation $(--+)$.

The analogous theorems take please with relevant changes for different combinations of variation. The following theorem corresponds to the case when there is a two-sided variation at the points t_{00}, τ_0 and t_{10}.

Theorem 3.5. *Let* ϑ_0 *be an optimal element, and* $x_0(t) = (p_0(t), q_0(t))^T$, $t \in [\widehat{\tau}, t_{10}]$ *is the corresponding solution. Let, besides, the functions* $u_0(t)$ *and* $u_0(t - \rho)$ *be continuous at the points* $t_{00}, t_{00} + \tau_0, t_{10}$. *Then, there exists a vector* $\pi = (\pi_0, \pi_1, \ldots, \pi_l), \pi_0 \le 0$ *and a solution* $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ *of equation* (3.5) *with initial condition* (3.6), *such that conditions* 3.4) $-$ 3.10) *are fulfilled. Moreover,*

$$\pi Z_{0t_0} + [\pi Z_{0q} + (\psi_{k+1}(t_{10}), \ldots, \psi_n(t_{10}))]\dot{g}_0(t_{00}) = \psi(t_{00}) f_0 + \psi(t_{00} + \tau_0) f_{01};$$
$$\pi Z_{0t_1} = -\psi(t_{10}) f_{01};$$
$$\pi Z_{0\tau} = \psi(t_{00} + \tau_0) f_{01} + \int_{t_{00}}^{t_{10}} \psi(t) f_p[t] \dot{p}_0(t - \tau_0) dt,$$

where

$$f_0 = f(t_{00}, x_0(t_{00}), \varphi_0(t_{00} - \tau_0), g_0(t_{00} - \sigma_0), u_0(t_{00}), u_0(t_{00} - \rho),$$

$$v_0(t_{00}), v_0(t_{00}-\theta_0)),$$

$$f_{01} = f(t_{00}+\tau_0, x_0(t_{00}+\tau_0), p_{00}, q_0(t_{00}+\tau_0-\sigma_0), u_0(t_{00}+\tau_0), u_0(t_{00}+\tau_0-\rho)), v_0(t_{00}+\tau_0), v_0(t_{00}+\tau_0-\theta_0) -f(t_{00}+\tau_0, x_0(t_{00}+\tau_0), p_{00}, q_0(t_{00}+\tau_0-\sigma_0), u_0(t_00+\tau_0),$$

$$u_0(t_{00}+\tau_0-\rho), v_0(t_{00}+\tau_0), v_0(t_{00}+\tau_0-\theta_0)$$

$$f_{02} = f(t_{10}, x_0(t_{10}), x_0(t_{10}-\tau_0), g_0(t_{10}-\sigma_0), u_0(t_{10}), u_0(t_{10}-\rho), v_0(t_{10}), v_0(t_{10}-\theta_0)).$$

We can establish theorems for different variants, which correspond to both-sided variation of the point t_{00} and one-sided variation of the rest of points, or for one-sided variation of the point t_{00} and to both-sided variation of the rest of them, etc.

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