Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 38, 2024

THE CLASS OF NON-INTEGER VERTICES FOR THE INITIAL RELAXATION POLYTOPE OF THE LINEAR ORDERING PROBLEM

George Bolotashvili

Abstract. The NP- hard linear ordering problem is solved as an integer linear programming problem. In this case, an important role is played by the study of the initial relaxation polytope for the polytope of the linear ordering problem. In this paper, we consider a non-integer point and we prove that this point is a non-integer vertex for the initial relaxation polytope of the linear ordering problem.

Keywords and phrases: NP hard problem, linear ordering problem, polytope, facets, polynomial algorithm.

AMS subject classification (2010): 68R05.

1 Introduction. Let $G = (N_n, E)$ be a complete orientation graph, where to each edge $(i, j) \in E$ corresponds a weight c_{ij}. Then, in the terminology of graphs, the linear ordering problem can be formulated as follows: Find an acyclic tournament of maximum weight in a complete edge-weighted digraph. If each acyclic tournament $i_1, \ i_2, \ \ldots, \ i_n$ is associated with a point in $n^2 - n$ dimensional space as follows:

$$x_{i_s i_q} = \begin{cases} 1, \ if \ s < q, \\ \\ 0, \ if \ s > q; \end{cases}$$

then of the linear ordering problem, as an integer linear programming problem, has the form:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \to max$$

$$0 \le x_{ij} \le 1, x_{ij} + x_{ji} = 1, i \ne j, \quad i, j = 1, \ \ldots, \ n; \tag{1}$$

$$0 \le x_{ij} + x_{jk} - x_{ik} \le 1, i \ne j, \quad i \ne k, \quad j \ne k, \quad i, j, k = 1, \ \ldots, \ n; \tag{2}$$

$$x_{ij} \in 0, 1, i \ne j, \quad i, j = 1, \ \ldots, \ n.$$

The polytope corresponding to the system (1), (2) will be denoted by B_n and will be called the initial relaxation polytope. The linear convex hull of the integer vertices of the polytope B_n will be called the linear ordering polytope and will be denoted by P_n. Taking into account the system of equalities $x_{ij} + x_{ji} = 1, i \ne j, \quad i, j = 1, \ \ldots, \ n$, polytopes B_n and P_n are considered in $\frac{n(n-1)}{2}$ dimensional space.

The number of facets of the polytope of the linear order problem is exponential, so when solving the problem it is necessary to study non-integer vertices for the initial relaxation polytope of the linear ordering problem. What will give us the opportunity to gradually add the necessary facets when solving a problem. In works [1,2], facets are constructed using non-integer vertices, and in works [3,4] the important properties of non-integer vertices for the initial relaxation polytope of the linear ordering problem are given.

2 The class of non-integer vertices for the polytope B_n. We provide a definition of a Mobius ladder with the help of which a point is constructed and it is proved that this point is non-integer vertices for the initial relaxation polytope of the linear ordering problem.

Definition 1. Let $D = (V, M)$ be a sub digraph of D_n which is generated by k dicycles $C_1, \ldots, C_k$ i. e. $V = \cup V(C_i), M = \cup C_i$, satisfies the following properties:

(a1) $k \geq 3$ and k is odd.

(a2) The length of C_i is three or four, $i = 1, \ldots, k$.

(a3) The degree of each node $u \in V(M)$ is at least three.

(a4) If two dicycles C_i and $C_j, 2 \leq i + 1 < j \leq k$ have a node, say v, in common then C_j is either left-adjacent or right-adjacent to C_i bat not both.

(a5) Given any dicycle $C_j, j \in \{1, \ldots, k\}$ set $J = \{1, \ldots, k\} \cap (\{j-1,\ \ j-3, \ldots, j-k+2\}\ \cup j+1,\ j+3, \ldots, k-2)$. Then the set $M - \{e_i, i \in J\}$ contains exactly one dicycle, namely C_j.

Then D is called a Mobius ladder.

In the Mobius ladder, we add one edge to all dicycles of length 3, so that we again obtained Mobius ladder, where all dicycles have length 4. Further, when examining the facets of the polytope P_n, which are described using Mobius ladder, from any Mobius ladder we can go to the Mobius ladder, where all dicycles have a length of 4, and then we can go back again. Mobius ladder, where all dicycles have a length of 4, is shown in Fig. 1.

In this work we will study Mobius ladder, where all dicycles have length 4 and with the help of these Mobius ladder, when removing common ribs, we get non-integer vertices for the initial relaxation polytope of the linear ordering problem.

In the future, in the paper we will also draw a directed graph corresponding to non-integer vertices of the polytope B_n, where the value along the edge direction is 1, along the reverse direction of the edge is 0, and the edges that will be absent in the figure are equal to 1/2 Fig. 2.

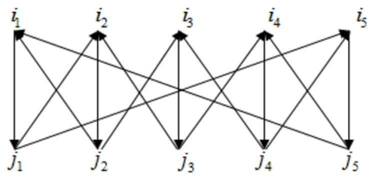

Figure 1: Mobius ladder.

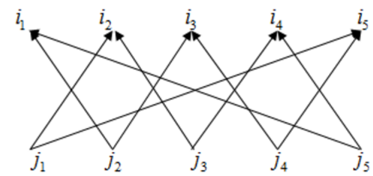

Figure 2: Non-integer point.

We introduce the notation $x_{ij}+x_{jk}-x_{ik}=(i,j,k), i\neq j,\quad i\neq k,\quad j\neq k,\quad i,j,k=1,\ \ldots,\ n.$

Theorem 1. *The non-integer point*

$$x^0=\left(x_{i_s i_l}=\frac{1}{2},\ x_{j_s j_l}=\frac{1}{2},\ s,l=1,\ldots,m;\ x_{i_s j_s}=\frac{1}{2},\ x_{j_s i_s}=\frac{1}{2},\ s=1,\ldots,m;\right.$$

$$x_{i_s j_{s+1}}=0,\ x_{i_s j_{s-1}}=0,\ x_{j_{s+1} i_s}=1,\ x_{j_{s-1} i_s}=1,\ s=1,\ldots,m;\ x_{i_s j_l}=\frac{1}{2},$$

$$\left. x_{j_l i_s}=\frac{1}{2},\ |s-l|\ \geq 2,\ s,l=1,\ldots,m\right),$$

where $(i_1,i_2,\ldots,\ i_m\ ),(j_1,j_2,\ldots,\ j_m\ ),m\geq 3,$ *are disjoint subsets from the set* $1,\ \ldots,\ n,$ *is the vertex of the polytope* B_n.

Proof. Let us consider some of the systems of linear inequalities from (1), (2), which turn into equality at the point x^0 :

$$\begin{array}{l}
x_{i_s j_{s+1}}=0,\ x_{i_s j_{s-1}}=0,\ s=1,\ldots,m;\\
(i_1,\ j_2,\ i_2)=0,\ (i_1,\ i_2,\ j_1)=0 \Longleftrightarrow (i_1,\ j_2,\ j_1)=0,\ (j_2,\ i_2,\ j_1)=0;\\
(i_2,\ j_3,\ i_3)=0,\ (i_2,\ i_3,\ j_2)=0 \Longleftrightarrow (i_2,\ j_3,\ j_2)=0,\ (j_3,\ i_3,\ j_2)=0;\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(i_{m-1},\ j_m,\ i_m)=0,\ (i_{m-1},\ i_m,\ j_{m-1})=0 \Longleftrightarrow (i_{m-1},\ j_m,\ j_{m-1})=0,\\
(j_m,\ i_m,\ j_{m-1})=0;\\
(i_m,\ j_1,\ i_1)=0,\ (i_m,\ i_1,\ j_m)=0 \Longleftrightarrow (i_m,\ j_1,\ j_m)=0,\ (j_1,\ i_1,\ j_m)=0.
\end{array} \tag{3}$$

From the last system of equalities from equalities of the second type in each column in the basis we leave three equalities. We also include equalities of the first type into the basis. Then, after a linear transformation along the columns, we obtain the corresponding diagonal matrix, where on the diagonal there is one two and the rest are ones. This matrix corresponds to the following unknowns: $x_{i_s j_{s+1}}, x_{i_s j_{s-1}}, s=1,\ldots,m; x_{i_q j_q},\quad q=1,\ldots,m.$

Next, we gradually add to system (3) an equality with only one new unknown, so that the vertex x^0 from Theorem 1 satisfies this equality. Therefore, after a linear transformation, we obtain the corresponding diagonal matrix.

- When adding equality $(i_k,i_{k+1},j_{k+2})=0$ unknown will be added $x_{i_k j_{k+2}}, k=1,,m;$
- When adding equality $(i_k,j_{k-1},i_{k+1})=0$ unknown will be added $x_{i_{k+1} j_{k-1}}, k=1,,m;$

- When adding equality $(i_k, i_{k+3}, j_{k+2}) = 0$ unknown will be added $x_{i_k i_{k+3}}, k = 1, , m$;
- When adding equality $(i_k, j_{k+1}, j_{k-2}) = 0$ unknown will be added $x_{j_{k-2} j_{k+1}}, k = 1, , m$;
- When adding equality $(i_k, i_{k+3+u}, j_{k+4+u}) = 0$ unknown will be added $x_{i_k j_{k+4+u}}, k = 1, , m, u = 0, , (m-1)/2 - 3$;
- When adding equality $(i_k, j_{k-1}, i_{k+3+u}) = 0$ unknown will be added $x_{i_{k+3+u} j_{k-1}}, k = 1, , m, u = 0, , (m-1)/2 - 3$;
- When adding equality $(i_k, i_{k+4+u}, j_{k+3+u}) = 0$ unknown will be added $x_{i_k i_{k+4+u}}, k = 1, , m, u = 0, , (m-1)/2 - 3$;
- When adding equality $(i_k, j_{k+1}, j_{k-3-u}) = 0$ unknown will be added $x_{j_{k-3-u} j_{k+1}}, k = 1, , m, u = 0, , (m-1)/2 - 3$;
- When adding equality $(i_k, i_{k+2}, j_{k-1}) = 0$ unknown will be added $x_{i_k i_{k+2}}, k = 1, , m$;
- When adding equality $(i_k, j_{k+1}, j_{k-1}) = 0$ unknown will be added $x_{j_{k-1} j_{k+1}}, k = 1, , m$;

Thus, for each unknown we wrote out a inequality and after a linear transformation we obtained the corresponding diagonal matrix. x^0 is also a valid point. The theorem is proven. $\square$

REFERENCES

1. Bolotashvili G., Kovalev M., Girlich E. New facets of the Linear ordering polytope. *SIAM J. Discrete Mathematics.*, **12**, 3 (1999), 326-336.

2. Bolotashvili, G. Solving the linear ordering problem using the facets (NP=P). *Bull. Georg. Natl. Acad. Sci.*, **17**, 1 (2023), 28-37.

3. Bolotashvili, G. The canonical relaxation polytope of the linear ordering problem. *Bull. Georg. Natl. Acad. Sci.*, **171**, 3 (2005), 445-448.

4. Bolotashvili, G. The examples of the non-integer vertices of the relaxation polytope of the linear ordering problem and their structure. *Institute of Applied Mathematics*, **20**, 1-3 (2005), **17**, 1-3 (2005), 28-32.

Received 20.05.2024; revised 22.07.2024; accepted 15.09.2024.

Author(s) address(es):

George Bolotashvili
V. Chavchanidze Institute of Cybernetics
Georgian Technical University
Zurab Anjaparidze str. 5, 0186 Tbilisi, Georgia
E-mail: bolotashvili@yahoo.com