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PRIME NUMBERS REPRESENTED BY BINARY FORMS WITH ODD DISCRIMINANT

Teimuraz Vepkhvadze

Abstract. The formulae for the average number of representations of positive integers by the genus of positive binary forms with an odd discriminant are obtained. It gives us the opportunity to characterize the primes which can be represented by binary forms with an odd discriminant.

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1 Introduction Let $f = ax^2 + bxy + cy^2$ be a primitive, integral, binary quadratic form of discriminant d, a, b, c are integers such that $gcd(a, b, c) = 1$, and $d = b^2 - 4ac$. The positive integer n is said to be represented by the form f if there exists $(x; y) \in Z \times Z$ such that $n = ax^2 + bxy + cy^2$. The number of representations of n by the form f is denoted by $r(n; f)$. In case of positive binary form, that is form f with $d < 0$, $r(n; f)$ is finite. The set of distinct classes of primitive, integral, binary quadratic form of discriminant d is denoted by $H(d)$. With respect to Gaussian composition (see for example, [1]), $H(d)$ is a finite Abelian group called the form class group. The order of $H(d)$ is called the form class number and is denoted by $h(d)$. There are quantities associated with a particular form, invariant in that they are equal for all integers represented by said form which separate the forms of given discriminant into general which may or may not coincide with the several classes.

These invariants the so-called characters, in case of an odd discriminant, are defined by

Theorem 1 (see, for example, [2]). *If $p_1, p_2, \ldots, p_k$ are the distinct odd prime factors of $\Delta = -d$, then $\left(\frac{n}{p_i}\right)$ has the same value for all positive integers n prime to Δ, represented by a form f of discriminant d.*

All forms of a given discriminant whose characters have the same value are said to form a genus. Since equivalent forms represent the same numbers, all forms in the same class are in the same genus. If the genus of the binary form f contains one class, then according to Newman Hall's theorem [2], the problem of obtaining exact formulas for $r(n; f)$ is solved completely. It follows from the results of [3] and [4] that half of the sum $\rho(n; f)$ of a generalized singular series that corresponds to a binary form f is equal to the average number of representations of a positive integer n by the genus containing this binary form. In particular, if a quadratic forms f belongs to one-class genus, then for positive integer n

$$r(n;f)=\frac{1}{2}\rho(n;f).$$

The function $\rho(n;f)$, in case of odd discriminant d, can be calculated as follows (see, [5], pp. 79, 80).

Theorem 2. *Let $f=ax^2+bxy+cy^2$ be a primitive positive binary quadratic form with odd discriminant $d=b^2-4ac$, $(a,d)=1$, $\Delta=-d=r^2\omega$ (ω is a square-free number), $n=2^\alpha m$ $(2\dagger m)$, $p^l||\Delta$, $p^\beta||n$, $u=\prod_{p|n,\,p\dagger 2\Delta}p^\beta$ (p is an odd prime number), then*

$$\rho(n;f)=\frac{\pi\chi_2\prod\limits_{p|\Delta,\,p>2}\chi_p\sum\limits_{v|u}\left(\frac{-\Delta}{v}\right)}{\Delta^{\frac{1}{2}}\prod\limits_{p|r,\,p>2}\left(1-\left(\frac{-\omega}{p}\right)\frac{1}{p}\right)L(1;-\omega)},$$

where

$$\begin{aligned}\chi_2&=3\ \ \text{if}\ \ 2|\alpha,\qquad \Delta\equiv 3\ (\mathrm{mod}\,8);\\&=0\ \ \text{if}\ \ 2\dagger\alpha,\quad \Delta\equiv 3\ (\mathrm{mod}\,8);\\&=\alpha+1\ \ \text{if}\qquad \Delta\equiv 7\ (\mathrm{mod}\,8);\end{aligned}$$

$$\begin{aligned}\chi_p&=\left(1+\left(\frac{p^{-\beta}na}{p}\right)\right)p^{\frac{1}{2}\beta}\quad \text{if}\ l\geq\beta+1,\ 2|\beta\\&=\left(1-\left(\frac{-p^{-l}\Delta}{p}\right)\frac{1}{p}\right)\left(1+\left(1+\left(\frac{-p^{-l}\Delta}{p}\right)\right)\frac{\beta-l}{2}\right)p^{\frac{1}{2}l}\ \ \text{if}\ l\leq\beta,\ 2|l,\ \ 2|\beta\\&=\left(1-\left(\frac{-p^{-l}\Delta}{p}\right)\frac{1}{p}\right)\left(1+\left(\frac{-p^{-l}\Delta}{p}\right)\right)\frac{\beta-l+1}{2}p^{\frac{1}{2}l}\ \ \text{if}\ l\leq\beta,\ 2|l,\ \ 2\dagger\beta\\&=\left(1+\left(\frac{p^{-l}\Delta}{p}\right)^{\beta+1}\left(\frac{p^{-(\beta+l)}na\Delta}{p}\right)\right)p^{\frac{1}{2}(l-1)}\ \ \text{if}\ l\leq\beta,\ \ 2\dagger l\\&=0\ \ \text{if}\ l\geq\beta+1,\ \ 2\dagger\beta.\end{aligned}$$

The values of $L(1;-\omega)$ are given in [6] (Lemma 15).

Using the Theorem 2, we can characterize primes which can be represented by the binary form of discriminant d for which class number $h(d)>1$. The case of discriminant d for which $h(d)=1$, was studied in [7]. For example, in case of discriminant $d=-11$, was obtained the following result: For the primes $p\neq 2,\,11:p=x^2+xy+3y^2$ if and only if $p\equiv 1,3,4,5,9\ (\mathrm{mod}\,11)$. Binary form $x^2+xy+3y^2$ is the only primitive form of discriminant $d=-11$.

Basic results. The set of binary forms with discriminant -15 splits into two genera, each consisting of one class with reduced forms, perspectively,

$$f_1=x^2+xy+4y^2\quad\text{and}\quad f_2=2x^2+xy+2y^2.$$

The corresponding representation functions can be calculated by using Theorem 2. It follows from this theorem, that any odd prime $p \neq 3$ and $p \neq 5$ is of the forms $x^2+xy+4y^2$ if and only if $\left(\frac{p}{3}\right) = 1$ and $\left(\frac{p}{5}\right) = 1$. Thus, the following theorem is proved.

Theorem 3. *For prime numbers* $p \neq 2, 3, 5 : p = x^2 + xy + 4y^2$ *if and only if* $p \equiv 1 \pmod{15}$, $p \equiv 4 \pmod{15}$.

Similarly, from the Theorem 2 we have, that any odd prime $p \neq 3$ and $p \neq 5$ is of the form $2x^2 + xy + 2y^2$ if and only if $\left(\frac{p}{3}\right) = 1$ and $\left(\frac{p}{5}\right) = 1$.

Thus, the theorem below is proved.

Theorem 4. *For primes* $p \neq 2, 3, 5 : p = 2x^2 + xy + 2y^2$ *if and only if* $p \equiv 2 \pmod{15}$, $p \equiv 8 \pmod{15}$.

The set of binary forms with discriminant -23 forms one genus, which consists of three classes with reduced forms

$$f_3 = x^2 + xy + 6y^2, \quad f_4 = 2x^2 + xy + 3y^2, \quad f_5 = 2x^2 - xy + 3y^2.$$

It is clear, that for any number $n, r(n; f_4) = r(n; f_5)$. It follows from the Theorem 2, that if any odd prime $p \neq 23$ is of the form $x^2 + xy + 6y^2$ or $2x^2 + xy + 3y^2$ then $\left(\frac{p}{23}\right) = 1$.

Thus, the theorem below is proved.

Theorem 5. *For primes* $p \neq 2, 23$, *if* $p = x^2 + xy + 6y^2$ *or* $p = 2x^2 + xy + 3y^2$ *then* $p \equiv 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 17, 18 \pmod{23}$.

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Author(s) address(es):

Teimuraz Vepkhvadze
Faculty of Exact and Natural Sciences
I. Javakhishvili Tbilisi State University
I. Chavchavadze Ave. 1, 0179 Tbilisi, Georgia
E-mail: t-vepkhbadze@hotmail.com