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TO THE APPLICATION OF THE MULTIPOINT METHOD

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Abstract. We report on the application of the multipoint method for numerical solution of two-dimensional boundary value problems for some linear and nonlinear differential equations of the elliptic type, using the continuous analogue of the alternating direction method. We also give the results of numerical realizations and comparisons with the Tikhonov-Samarskii and Volkov methods.

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Let us consider the boundary value problem for a linear strong-elliptic, self-adjoint system of differential equation in a domain D with boundary ∂D

$$Lu = \left(k^{ij}\partial_{ij} - q\right) u\left(x_1, x_2\right) = -f\left(x_1, x_2\right), \quad \left(x_1, x_2\right) \in D, \tag{1}$$

$$\left[u + \sigma\left(x_1, x_2\right) N(u)\right]_{\partial D} = \varphi\left(x_1, x_2\right), \tag{2}$$

where $k^{ij} = \left\|k_{\alpha\beta}^{ij}\left(x_1, x_2\right)\right\|_{\alpha,\beta=1}^{n}$ is a positive definite symmetrical matrix with constant coefficients

$$N(u) = \left\|N_\alpha(u)\delta_{\alpha\beta}\right\|_1^n, \quad N_\alpha(u) = \sum_{\beta=1}^{n} k_{\alpha\beta}^{ij} u_{\beta,i} \cos\left(\nu, x_j\right),$$

$\left\|\delta_{\alpha\beta}\right\|_{\alpha,\beta=1}^{n}$ is an identity matrix, ν is an external normal, $q = \left\{q_1\left(x_1, x_2\right), \ldots, q_n\left(x_1, x_2\right)\right\}$ is a diagonal nonnegative matrix, $\sigma\left(x_1, x_2\right) > 0$ is a scalar function, $u = \left(u_1, u_2, \ldots, u_n\right)^T$ is an unknown, $f = \left(f_1, f_2, \ldots, f_n\right)^T$, and $\varphi = \left(\varphi_1, \ldots, \varphi_n\right)^T$ are known vector functions. The continuous analogue of the alternating direction iteration scheme for the boundary value problem (1) and (2), is defined by the following two series of expressions (see [1]):

$$\begin{aligned} B_1 u^{s+1/2} &= B_{12} u^s + f = F_{s+1/2}, \quad u^{s+1/2} + \sigma_1 N\left(u^{s+1/2}\right)\big|_{\partial_1 D} = \varphi_1, \\ B_2 u^{s+1} &= rIu^{s+1/2} + \left(B_2 - rI\right) u^s = F_{s+1}, \quad u^{s+1} + \sigma_2 N\left(u^{s+1}\right)\big|_{\partial_2 D} = \varphi_2\left(x_1\right), \end{aligned} \tag{3}$$

where r is an iterative parameter, I is an unit operator, $\sigma = \sigma_1 + \sigma_2$,

$$B_i = \left\|\left(r + \hat{q}^i\right) I + \delta_{\alpha\beta}\hat{A}_i\right\|_{\alpha,\beta=1}^{n}, \quad B_{12} = B_1 + L, \quad q^1 + q^2 = q,$$

$$\hat{q}^i \geq q_\alpha^i\left(x_1, x_2\right), \quad \hat{A}_i = -a_i\partial_{ii}, \quad a_i > 0.$$

The following theorem of convergence of the iterative process (3) is true.

Theorem 1. *Let the following conditions be fulfilled:*

i) Domain D *is a square,* $0 \le x_1, x_2 \le 1,\ A_i = -\partial_{ii},\ B = BL,$

ii) $bB \le -L < \frac{1}{r} B_1 B_2, \quad (b > 0, 0 < r \le 1), \quad B = (A_1 + A_2)\, \|\delta_{\alpha\beta}\|_1^n,$

iii) $u_\alpha^0 \in C^4(0,1;0,1) \quad (\alpha = \overline{1,n}).$

Then the sequence of vector functions u^s *converges to the solution* $u(x,y)$.

Below we consider some particular cases of this theorem when $N(u) = 0$.

Corollary 1. *In the case of differential equations of the plane theory of elasticity, the condition ii) is holds if we suppose that*

$$b = \mu, \quad B = (A_1 + A_2)\, \|\delta_{\alpha\beta}\|_1^2, \quad B_i = (rI + (2\lambda + 3\mu) A_i), \quad r \in (0,1).$$

Corollary 2. *In the case when* L *is the operator of Vekua's shell theory [1968], ii) is true, when (compare with subsection 12.1, see (12.9))*

$$b = \min\left\{1 - \varepsilon, \frac{(1-\eta)h^2}{2}\right\}, \quad B = (A_1 + A_2)\, \|\delta_{\alpha\beta}\|_1^5,$$
$$B_\alpha = B_\alpha\left((r + 2\mu)I + (2\lambda^* + 3\mu)\, A_\alpha\right), \quad r \in (0,1),$$

where $0 < \eta < 1,\ (1 + 2h^2\eta)^{-1} < \varepsilon < 1,$ *and* $2h$ *is the thickness of the shell.*

Let us consider the following differential equations

$$Lu = \partial_1 (k_1 \partial_1 u) + \partial_2 (k_2 \partial_2 u) - qu = f, \quad k_1,\ k_2 > 0, \quad q \ge 0.$$

Instead of the iterative process (3) the following scheme can be used:

$$\begin{aligned} &((r + q_1)\, I + A_1)\, u^{s+1/2} = (rI - A_2)\, u_s + f = F_{s+1/2}, \ \left[u^{s+1/2} + \sigma_1 \frac{\partial u^{s+1/2}}{\partial \nu}\right]_{\partial_1 D} = \varphi_1, \\ &((r + q_2)\, I + A_2)\, u^{s+1} = (rI - A_1)\, u_{s+1/2} + f = F_{s+1}, \ \left[u^{s+1} + \sigma_2 \frac{\partial u^{s+1}}{\partial \nu}\right]_{\partial_2 D} = \varphi_2, \end{aligned} \tag{4}$$

where $A_\alpha = -\partial_\alpha (k_\alpha \partial_\alpha)$.

Let $L = \partial_{\alpha\alpha},\ A_\alpha = -\partial_{\alpha\alpha}$. In this case, if the scheme (4) is applied, the following theorem is true.

Theorem 2. *Let the following conditions be fulfilled:*

i) $\|A_2 u\| < +\infty,$

ii) $0 < r < \pi^2.$

Then, for $\forall \varepsilon > 0\ \exists S(\varepsilon)$ *such that when* $s > S(\varepsilon)$ $\|u - u^s\| < \varepsilon.$

We remark that the alternating directions scheme (4), when ($L = \Delta$), after the substitution

$$A = \{I + \tau A_1, I + \tau A_2\}, \quad B = \{A_1 + A_2, A_1 + A_2\}, \quad u = \left(u^s, u^{s+1/2}\right)^T,$$
$$u_\tau = \left(\tau^{-1}\left(u^{s+1/2} - u^s\right), \tau^{-1}\left(u^{s+1} - u^{s-1/2}\right)\right)^T, \quad \varphi = \left(F_{s+1/2}, F_{s+1}\right)^T, \quad \tau = \frac{1}{r},$$

has been reduced to the canonical form of Samarski [2] for the operator equations $Au_\tau = Bu + \varphi$ in two-layer iteration processes, where the operators A and B are energetically equivalent. In the above theorem, this condition is not fulfilled.

Let us consider the problem of approximately solving BVP (1) and (2) using the iteration process (3) and (4). Correspondingly, a one dimensional BVP will be solved using the methods of Tikhonov, Samarskii [3], Volkov [4], or generalized factorization (see [5]) (GF) methods. We note that even though [3] is of an arbitrary order of accuracy, difficulties may arise from the coefficients on which [3] scheme is dependent, due to the necessity of computing recurrent integrals of the Volterra type. Volkov's approach is based on differentiating the given ODE to obtain the corresponding solution, however this process may be divergent. Thus, using the (GF) method, the aforementioned processes define the alternating direction method of an arbitrary order of accuracy.

Below, we provide numerical results obtained by solving PDEs using the alternating direction method (ADM) corresponding to the iteration schemes (4). The results are given in the corresponding tables. The tables below show the maximum absolute error between the function and its approximation at the n-th iteration, where $s = 3$ is the parameter of (P) and (Q) building blocks of the (GF) method.

For the linear case,

$$(\partial_1(k_1\partial_1) + \partial_2(k_2\partial_2) - q)\,u = -f(x_1, x_2), \quad \Gamma = I \times I, \quad I = (0, 1),$$

$$\partial_1(k_1\partial_1)\,u_{s+1} = -F_{s+1/2} + (q_1 + r)\,u_{s+1/2}, \quad \partial_2(k_2\partial_2)\,u_{s+1} = -F_{s+1} + (q_2 + r)\,u_{s+1},$$

$$k_1 = 1 + x^2y^2, \quad k_2 = 1 + x^5y^5, \quad q_1 = q_2 = x^4y^3, \quad u = x^6 + y^5 + x^5y^7,$$

$$h = 1/50, \quad r = 10, \quad s = 3, \quad u_0 = 0, \quad \partial_1(k_1\partial_1)\,u_0 = 0, \quad \partial_2(k_2\partial_2)\,u_0 = 0.$$

n	$\delta u^{s+1/2}$	δu^{s+1}
700	0.00011357157392666295	0.00011303945386575975
1000	3.125228458644713e-5	3.003011986635329 e-5
2000	1.108921056314216 e-6	1.3168026149479317 e-6
3000	1.0756186812344026 e-6	1.0858063368424098 e–6

Table 1: Max absolute errors for the linear case of (ADM)

For the nonlinear case,

$$u_{xx}+u_{yy}-u\left(u_x+u_y\right)=f\left(x_1,x_2\right),\quad \Gamma=I\times I,\quad I=(0,1),$$

$$\left(-r_1-r_2\partial_x+\partial_{xx}\right)u_{s+1/2}=-\left(r_1+r_2\partial_x\right)u^{[s]}+f+u^{[s]}u^{[s]}{}_x+u^{[s]}u^{[s]}{}_y=F_{s+1/2},$$

$$\left(-r_3-r_4\partial_y+\partial_{yy}\right)u_{s+1}=-r_3u^{[s+1/2]}-r_4u_y^{[s]}+f+u^{[s+1/2]}u_x^{[s+1/2]}+u^{[s+1/2]}u_y^{[s]}=F_{s+1},$$

$$u^0=0,\quad u_x^0=0,\quad u_y^0=0,\quad \partial_{11}u^0=0,\quad \partial_{22}u^0=0,\quad u=x^4+y^4+x^2y^2$$

$$r_1=6,\quad r_2=0.3,\quad r_3=6,\quad r_4=0.3,\quad h=1/90,\quad k=15,\quad s=3$$

n	$\delta u^{s+1/2}$	δu^{s+1}
50	0.04175659811124133	0.03578427496565073
200	0.014437167228944503	0.012164445108532096
400	0.00836032923218566	0.007038268105019618
800	0.004736168566869647	0.004044056365913651

Table 2: Max absolute errors for the nonlinear case containing divergence (ADM)

One-dimensional problems were solved using the (GF) method in the linear case, while additional iterations were applied to obtain results in the nonlinear case.

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