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ON THE APPLICATION OF COMPLEX ANALYSIS FOR THE ESSENTIALLY NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

Tamaz Vashakmadze

Abstract. If we consider the mathematical models, corresponding to refined theories for elastic plates, the main part of relevant differential operator contains together with Laplacian and biharmonic operator also a composition of Laplacian on Monge-Ampére nonlinear form of second degree and Poisson brackets. By using complex analysis, we construct a system of integro-differential equations the solution of which we find by Seidel method of successive approximation.

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To demonstrate development of methods of complex analysis, for essentially non-linear systems of partial DEs, it is sufficient to consider the following non-linear systems of PDEs (see [1, Ch. 1]) for an isotropic elastic plate with constant thickness

$$D\Delta^2\bar{u}_3=\left(1-\frac{h^2(1+2\gamma)(2-v)}{3(1-v)}\Delta\right)\left(g_3^+-g_3^-\right)+2h\left(1-\frac{h^2(1+2\gamma)}{3(1-v)}\Delta\right)[\bar{u}_3,\Phi^*]$$
$$+h\left(g^+_{\alpha,\alpha}+g^-_{\alpha,\alpha}\right)-\int_{-h}^{h}\left(zf_{\alpha,\alpha}-\left(1-\frac{1}{1-v}\Delta\left(h^2-z^2\right)f_3\right)\right)dz+R_3\left[\bar{u}_3;\gamma\right], \tag{1}$$

$$\Delta^2\Phi^*=-\frac{E}{2}\left[\bar{u}_3,\bar{u}_3\right]+\frac{v}{2}\Delta\left(g_3^++g_3^-\right)+\frac{1+v}{2h}\bar{f}_{a,a}+R_6\left[\Phi^*\right], \tag{2}$$

$$Q_{\alpha 3}-\frac{1+2\gamma}{3}h^2\Delta Q_{\alpha 3}=-D\Delta\bar{u}_{3,\alpha}+\frac{h^2(1+2\gamma)}{3(1-v)}\partial_\alpha\left(g_3^+-g_3^-+2h(1+v)\right)[\bar{u}_3,\Phi^*]$$
$$+h\left(g^+_\alpha+g^-_\alpha\right)-\int_{-h}^{h}zf_\alpha dz+\frac{1+v}{2(1-v)}\int_{-h}^{h}\left(h^2-z^2\right)f_{3,\alpha}dz+R_{3+\alpha}\left[Q_{\alpha 3};\gamma\right]. \tag{3}$$

Systems (1)–(3) without the remainder terms R yield 2D systems of refined theories with control parameters γ. By choosing the concrete value of γ we can get any of the existing refined theories, while for other values of γ these mathematical models are new. Here E is Young's modulus of elasticity and ν is Poisson's ratio, λ and μ are the Lamé coefficients. For unity, together with above system consider (see [2])

$$\begin{aligned}
&(\lambda^*+2\mu)\,\partial_1\tau+\mu\partial_2\omega=\frac{1}{2h}f_1+\mu\left(\partial_1\left(\bar{u}_{3,2}\right)^2-\partial_2\left(u_{3,1}u_{3,2}\right)\right)-\lambda_1\left(\sigma_{33,1},1\right),\\
&(\lambda^*-2\mu)\,\partial_2\tau-\mu\partial_1\omega=\frac{1}{2h}f_2-\mu\left(\partial_2\left(\bar{u}_{3,1}\right)^2-\partial_1\left(u_{3,1}u_{3,2}\right)\right)-\lambda_1\left(\sigma_{33,2},1\right),
\end{aligned} \tag{4}$$

where $\lambda^* = 2\lambda\mu(\lambda+2\mu)^{-1}$, $\lambda_1 = \frac{\lambda}{2h(\lambda+2\mu)}$ and the functions $\tau = \bar{\varepsilon}_{\alpha\alpha}$ and $\omega = \bar{u}_{1,2} - \bar{u}_{2,1}$ correspond to plane expansion and rotation. By (4), the second equation with respect to Airy's function in the von *Kármán* system takes the form

$$(\lambda^* + 2\mu)\,\Delta\bar{\varepsilon}_{\alpha\alpha} = \frac{(\lambda^\star + 2\mu)\,(3\lambda + 2\mu - 2)}{2\mu(3\lambda + 2\mu)}\Delta\left(\sigma_{11} + \sigma_{22}\right) = \frac{1}{2h}\bar{f}_{\alpha,\alpha}$$

$$+\mu\left(\partial_{11}\left(\bar{u}_{3,2}\right)^2 - 2\partial_{12}\left(\bar{u}_{3,1}\bar{u}_{3,2}\right) + \partial_{22}\left(\bar{u}_{3,1}\right)^2\right) + \frac{1}{2h}\left(\frac{\lambda\left(\lambda^* + 2\mu\right)}{2\mu(3\lambda + 2\mu)} - \frac{\lambda}{\lambda + 2\mu}\right)\int_{-h}^{h}\Delta\sigma_{33}dz,$$

or

$$\Delta\left(\bar{\sigma}_{11} + \bar{\sigma}_{22}\right) = -\frac{E}{2}\left[\bar{u}_3, \bar{u}_3\right] + \frac{v}{2h}\int_{-h}^{h}\Delta\sigma_{33}dz + \frac{1+v}{2h}\bar{f}_{\alpha,\alpha}. \tag{5}$$

For ω we get

$$\Delta\omega = \frac{1}{2h\mu}(f_{1,2} - f_{2,1}) + \partial_{12}(u_{3,2}^2 - u_{3,1}^2) \text{ or } \Delta\omega = \frac{1}{2h\mu}(f_{1,2} - f_{2,1}) + \{u_3, (\partial_{22} - \partial_{11})u_3\}. \tag{6}$$

Here we note that in the right hand side of equation (6) we have well-known Poisson brackets

$$\{f, g\} = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right),$$

in the case of $f = u_3$, $g = (\partial_{22} - \partial_{11})u_3$, $p_1 = x_1$, $q_1 = x_2$, $i = j = N = 1$.

The calculation and analysis of the symbolic determinant of systems of type (1)–(3) show that the characteristic forms of such systems may be positive, negative or zero because they represent arbitrary functions of x, y. Consider the expression

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad u(x,y) = U(z,\bar{z}), \quad \partial_1 = \partial/\partial x_1, \quad \partial_2 = \partial/\partial x_2,$$

$$\partial_{\bar{z}} = \tfrac{1}{2}\left(\partial_1 + i\partial_2\right), \quad \partial_z = \tfrac{1}{2}\left(\partial_1 - i\partial_2\right), \quad 4\partial_{z\bar{z}} = \Delta,$$

$$16\partial_{z\bar{z}}\partial_{\bar{z}z} = \Delta^2, \quad -16\partial_{\bar{z}z}[U(\bar{z},z), V(\bar{z},z)] = \Delta[u(x,y), v(x,y)].$$

Now we can formulate the iterative-direct method by means of which solutions of the DEs (3), (4) can be found if they are rewritten in complex variables.

Let $L[U(\bar{z},z)]^{[m]}$ denote the m-th iteration for the deflection $u_3^*(x,y)$, which is calculated by the known right-hand terms (without R) and the $(m-1)$-th iteration of the summand

$$\frac{2Eh}{16D}\left(1 - \frac{h^2(1+2\gamma)}{3(1-v)}\partial_{\bar{z}}\partial_z\right)\int_0^{\bar{z}}\int_0^{z}(\bar{z}-\bar{\zeta})(z-\zeta)[U,V]^{[m-1]}d\bar{\zeta}d\zeta,\; EV = \Phi\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right). \tag{7}$$

We do analogous operations for the shearing forces in DEs (3) as well as for system (4). Let

$$V^{[m]} = V^{[m]}(\bar{z},z) = -\frac{\mu}{\lambda^* + 2\mu}\int_0^{\bar{z}}\int_0^{z}(\bar{z}-\bar{\zeta})(z-\zeta)\left[U^{[m-1]}, U^{[m-1]}\right]d\bar{\zeta}d\zeta + F(\bar{z},z). \tag{8}$$

Thus, by applying complex analysis, we have reduced the systems of PDEs of KMR type to the pseudointegral operator of second type.

The iterative scheme described by (7) corresponds to the solution of a Volterra second type non-linear integral equation, whereas the processes described by the schemes generated from (1) contain both Volterra and Fredholm type operators with an arbitrary parameter γ. The convergence of a Volterra type process (where $\gamma = -0.5$) depends also on a proper selection of the initial functions $U^{[0]}$ and $V^{[0]}$. We can apply some results of [4, XXIV, p. 476, Example 4] to the equation $[u, u] + a^2 = 0$ for an arbitrary function $a = a(x, y)$. When $\gamma \neq -0.5$, the convergence depends also on the Fredholm type operator

$$F_r(U, V) = \lambda \partial_{\bar{z}} \partial_z \int_0^{\bar{z}} \int_0^z (\bar{z} - \bar{\zeta})(z - \zeta)[U(\bar{\zeta}, \zeta), V(\bar{\zeta}, \zeta)] d\bar{\zeta} d\zeta,$$

with an arbitrary parameter denoted by λ. The operator $\lambda^{-1} F(U, V)$ depends on the behavior of expressions which may generate different kinds of wave (shock, soliton) functions, and if they are uniformly bounded functions, then the processes corresponding to applications of the above Fredholm type operator will be convergent since the corresponding operator will be the contracted one. Consider the following differential equation of complex variable functions

$$\begin{gathered} \partial^2_{\bar{z}z} \Psi = -4\,[U, U] + \frac{1}{2}\,\{U, -4\partial_{\bar{z}z} U_3\} + \tilde{F}(\bar{z}, z), \quad \Psi(\bar{z}, z) = (\lambda + 2\mu)\mathrm{T}(\bar{z}, z) + \mu\Omega(\bar{z}, z), \\ \tau(x, y) = \mathrm{T}(\bar{z}, z), \quad \omega(x, y) = \Omega(\bar{z}, z), \quad \sigma_{33}(x, y) = \Sigma_{33}(\bar{z}, z), \qquad (9) \\ \tilde{F}(\bar{z}, z) = \frac{1}{4h}\,(F_{\alpha,\alpha}(\bar{z}, z) + F_{1,2} - F_{2,1}) + 4\lambda_1\,(\partial_{\bar{z}z}\Sigma_{33}, 1)\,, \quad f_\alpha(x, y) = F(\bar{z}, z). \end{gathered}$$

Using this approach, we complete [3, p. 15, Theorem 8]. The following statement is true.

Theorem 1. *If we consider the following Seidel iterative schemes corresponding to bending and expansion-compression* (1)–(3) *and* (9) *processes, given by the following expressions*

$$\begin{gathered} V^{[m]}(\bar{z}, z) = a \int_0^{\bar{z}} \int_0^z (\bar{z} - \bar{\zeta})(z - \zeta) \left[U^{[m-1]}, U^{[m-1]}\right] d\bar{\zeta} d\zeta, \quad m = 1, 2, \dots, \\ U_3^{[m]}(\bar{z}, z) = b \int_0^{\bar{z}} \int_0^z (\bar{z} - \bar{\zeta})(z - \zeta) \left[U^{[m-1]}, V^{[m]}\right] d\bar{\zeta} d\zeta + c \int_0^{\bar{z}} \int_0^z \left[U^{[m-1]}, V^{[m]}\right] d\bar{\zeta} d\zeta, \\ \Psi^{[m]} = \int_0^{\bar{z}} \int_0^z d\bar{\zeta} d\zeta \left(4\left[U^{[m-1]}, U^{[m-1]}\right] + \frac{1}{2}\left\{U^{[m-1]}, -4\partial_{\bar{z}z} U^{[m-1]}\right\}\right), \end{gathered}$$

where $a = -\frac{1}{2}(1 - \nu)$, $b = \frac{3}{16h^2}(1 - \nu^2)$, *and* $c = -\frac{(1+2\gamma)(1+\nu)}{16}$, *then the corresponding schemes are convergent in an open domain, when* $|\bar{z}|, |z| < 1$.

Proof. For different powers of $\bar{z}$ and z we compute

$$\begin{gathered} [\bar{z}^r z^s, \bar{z}^r z^s] = -2rs(r + s - 1)\bar{z}^{2r-2} z^{2s-2}, \quad V^{[m]} = -\frac{a}{2} \frac{(r + s - 1)}{(2s - 1)(2r - 1)} \bar{z}^{2r} z^{2s}, \\ \left[\bar{z}^r z^s, \bar{z}^{2r} z^{2s}\right] = -2rs(3(s + r) - 2)\bar{z}^{3r-2} z^{3s-2}, \\ c \int_0^z \int_0^{\bar{z}} \left[U^{[m-1]}, V^{[m]}\right] d\bar{\xi} d\xi = ac \frac{(s + r - 1)(3(s + r) - 2)rs}{(2s - 1)(3s - 1)(2r - 1)(3r - 1)} \bar{z}^{3r-1} z^{3s-1}, \end{gathered}$$

and

$$b\int_0^z\int_0^{\bar z}(\bar z-\bar\xi)(z-\xi)\left[U_3^{[m-1]},V^{[m]}\right]d\bar\xi d\xi=\frac{ab}{9}\frac{(s+r-1)(3(s+r)-2)}{(2s-1)(3s-1)(2r-1)(3r-1)}\bar z^{3r}z^{3s},$$

for Poisson brackets we get

$$\{U,-4\partial_{\bar z z}U\}=-8i[U_z\partial_{\bar z\bar z}U_3-U_{\bar z}\partial_{zz}U]\quad\text{or}\quad\{U,-4\partial_{\bar z z}U\}=-4i[\partial_{\bar z\bar z}U_z^2-\partial_{zz}U_{\bar z}^2],$$

thus,

$$\left\{U^{[m-1]},-4\partial_{\bar z z}U^{[m-1]}\right\}=-4i[2rs(s-r)\bar z^{2r-2}z^{2s-2}],$$

consequently,

$$\int_0^z\int_0^{\bar z}\left\{U^{[m-1]},-4\partial_{\bar z z}U^{[m-1]}\right\}d\bar\xi d\xi=i[8rs\frac{s-r}{(2r-1)(2s-1)}\bar z^{2r-1}z^{2s-1}].$$

As powers of $\bar z$ and z grow on each iteration with a factor no less than about two, it is easy to show that $U_3^{[m]}$, $\Psi^{[m]}$, and $V^{[m]}$ are fundamental sequences of functions and this gives us convergence, because $\bar z^r z^s$ are basis of space of complex functions. □

Remark 1. For the summands from system (1)–(4) containing $(\Delta\sigma_{33},1)$, we must use the Euler–Maclaurin quadrature formula. Closeness with the classical theory can be demonstrated by considering the Pompeiu formula [5, (4.11) or (4.13); I, 4], then we immediately obtain an explicit solution of the Cauchy–Riemann inhomogeneous system of DEs.

$$w(\bar z,z)=(\lambda+2\mu)\tau\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right)+i\mu\omega\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2i}\right).$$

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Author(s) address(es):

Tamaz Vashakmadze
I.Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University
University str. 11, 0186 Tbilisi, Georgia
E-mail: tamazvashakmadze@gmail.com