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SOLUTION OF THE CAUCHY PROBLEM IN QUADRATURES FOR ONE CLASS OF STRICTLY HYPERBOLIC EQUATIONS OF HIGH ORDER

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Abstract. The paper proposes an approach that allows one in a constructive way to effectively write out the solution to the Cauchy problem in quadratures for one class of strictly hyperbolic equations of high order.

Keywords and phrases: Higher order hyperbolic equations, general solution, Cauchy problem.

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Consider the strictly hyperbolic equation on the following form [1]:

$$Lu := \frac{\partial^n u}{\partial t^n} + a_{n-1}\,\frac{\partial^n u}{\partial t^{n-1}\partial x} + a_{n-2}\,\frac{\partial^n u}{\partial t^{n-2}\partial x^2} + \cdots + a_1\,\frac{\partial^n u}{\partial t\partial x^{n-1}} + a_0\,\frac{\partial^n u}{\partial x^n} = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function of two variables x and t, a_i, $i = 0, \dots, n-1$, are given constant coefficients, $n \geq 2$. In this case, due to the strictly hyperbolicy of equation (1), the corresponding characteristic equation

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0$$

has only simple real roots $\lambda_1, \dots, \lambda_n$. This in turn implies that

$$P(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$$

and in accordance with this, equation (1) can be rewritten in the following form

$$Lu := \prod_{i=1}^{n}\Big(\frac{\partial}{\partial t} - \lambda_i\,\frac{\partial}{\partial x}\Big)u = 0. \tag{2}$$

As is known, the general solution to equation (2) is given by the following formula [2]:

$$u(x,t) = \sum_{i=1}^{n} \tau_i(x + \lambda_i t), \tag{3}$$

where $\tau_i = \tau_i(s)$, $s \in \mathbb{R}$, $i = 1, \dots, n$, are arbitrary functions of class $C^n(\mathbb{R})$. For equation (2), consider the Cauchy problem with the following initial conditions

$$\frac{\partial^i u}{\partial t^i}\,(x,0) = \varphi_i(x), \quad i = 0, \dots, n-1, \tag{4}$$

where $\varphi_i = \varphi_i(x)$, $i = 0, \dots, n-1$, are given functions of class $C^{n-i}(\mathbb{R})$.

Remark 1. Note that solving the Cauchy problem (2), (4) in quadratures, using representation (3) of the general solution already in the case $n = 3$ encounters certain difficulties and these difficulties increase sharply as n increases. In order to get around these difficulties and taking into account that equation (2) is linear, we look for a solution to the Cauchy problem (2), (4) in the following form

$$u(x,t) = \sum_{i=1}^{n} u_i(x,t), \tag{5}$$

where $u_i(x,t)$ is the solution to the following Cauchy problem

$$Lu_i = \prod_{j=1}^{n} \Big(\frac{\partial}{\partial t} - \lambda_j \frac{\partial}{\partial x}\Big) u_i = 0, \tag{6}$$

$$\frac{\partial^{i-1} u_i}{\partial t^{i-1}}(x,0) = \varphi_{i-1}(x), \quad \frac{\partial^{j} u_i}{\partial t^{j}}(x,0) = 0, \ \ j \neq i-1, \ \ j = 0, \dots, n-1. \tag{7}$$

Moreover, taking into account the structure (3) of the general solution, we look for the solution itself to the Cauchy problem (6), (7) on the following form

$$u_i(x,t) = \sum_{j=1}^{n} \alpha_{ij} (\mathcal{J}^{i-1}\varphi_{i-1})(x + \lambda_j t), \tag{8}$$

where α_{ij} are constants to be determined, and $\mathcal{J}^{i-1}$, is an operator that, for $i > 1$, acts according to the following formula

$$(\mathcal{J}^{i-1}\varphi_{i-1})(x) = \frac{1}{(i-2)!} \int\limits_0^x (x-\tau)^{i-2} \varphi_{i-1}(\tau)\, d\tau \tag{9}$$

and represents an antiderivative of order $i-1$, and when $i = 1$ it is assumed that

$$i = 1: \ (\mathcal{J}^{i-1}\varphi_0)(x) = \varphi_0(x). \tag{10}$$

It is proved that the constants α_{ij} included in representation (8) are given by the formulas

$$\alpha_{ij} = \frac{\Delta_{ij}}{\Delta}, \ \ i,j = 1, \dots, n, \tag{11}$$

where

$$\Delta = \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \hdotsfor{3} \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = \prod_{i<j} (\lambda_j - \lambda_i) \neq 0$$

is the Vandermonde determinant, and the determinant Δ_{ij} is obtained from the determinant Δ by replacing the j-th column with a column whose all elements are equal to zero, except for the i-th element, which is equal to one.

Thus, if we take into account that to a solution of the problem $(2),(4)$ uniqueness theorem holds [1], then the validity of the following theorem will follow from (5), (8)–(11).

Theorem. *The unique solution to the Cauchy problem* $(2),(4)$ *is given by the formula*

$$u(x,t)=\sum_{i=1}^{n}u_i(x,t),$$

where

$$u_1(x,t)=\sum_{j=1}^{n}\alpha_{1j}\varphi_0(x+\lambda_j t),$$

$$u_i(x,t)=\frac{1}{(i-2)!}\sum_{j=1}^{n}\alpha_{ij}\int\limits_{0}^{x+\lambda_j t}(x+\lambda_i t-\tau)^{i-2}\varphi_{i-1}(\tau)\,d\tau,\quad i=2,\dots,n,$$

and the constants α_{ij} *are given by equalities* (11).

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