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ON THE SPACE OF GENERALIZED THETA-SERIES WITH SPHERICAL POLYNOMIALS OF FOURTH ORDER

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Abstract. The spherical polynomials of order $\nu = 4$ with respect to some diagonal quadratic form of five variables are constructed and the basis of the space of these spherical polynomials is established. The space of generalized theta-series with respect to some quadratic form of five variables is considered and the basis of this space is constructed.

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Introduction. Let

$$Q(X) = Q(x_1, x_2, \ldots, x_r) = \sum_{1 \le i \le j \le r} b_{ij} x_i x_j$$

be an integer positive definite quadratic form of r variables and let $A = (a_{ij})$ be the symmetric $r \times r$ matrix of the quadratic form $Q(X)$, where $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, for $i < j$. If $X = (x_1 \ldots x_r)^T$ denotes a column matrix and X^T its transpose, then $Q(X) = \frac{1}{2} X^T A X$. Let A_{ij} denote the cofactor to the element a_{ij} in A and a^*_{ij} is the element of the inverse matrix A^{-1}.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1 \le i,j \le r} a^*_{ij} \Big(\frac{\partial^2 P}{\partial x_i \partial x_j} \Big) = 0$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [1]), and

$$\vartheta(\tau, P, Q) = \sum_{n \in \mathbb{Z}^r} P(n) z^{Q(n)}, \qquad z = e^{2\pi i \tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im} \tau > 0 \tag{1}$$

is the corresponding generalized r-fold theta-series.

Let $\mathcal{P}(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$.

Hecke [2] calculated the dimension of the space $\mathcal{P}(\nu, Q)$ and showed that

$$\dim \mathcal{P}(\nu, Q) = \binom{\nu + r - 1}{r - 1} - \binom{\nu + r - 3}{r - 1}.$$

He formed a basis of the space of spherical polynomials of second order ($\nu = 2$) with respect to $Q(X)$.

Lomadze ([3], p.533) proved that among homogenous polynomials of fourth order ($\nu = 4$) in r variables

$$\varphi_{ijkl} = x_i x_j x_k x_l \frac{1}{(r+4)D}(A_{ij}x_k x_l + A_{ik}x_j x_l + A_{il}x_k x_j + A_{jk}x_i x_l + A_{jl}x_k x_i \\ + A_{kl}x_i x_j)2Q + \frac{1}{(r+2)(r+4)D^2}(A_{ij}A_{kl} + A_{ik}A_{jl} + A_{il}A_{jk})(2Q)^2, \\ i,j,k,l = 1,\cdots,r \tag{2}$$

exactly $\frac{1}{24}r(r^2-1)(r+6)$ ones are linearly indeendent and form the basis of the space of spherical polynomials of fourth order with respect to $Q(x)$.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \; : \; P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [1] calculated the dimension of the vector space $T(\nu, Q)$ for reduced binary quadratic form Q and obtained an upper bound of the dimension of the space $T(\nu, Q)$ for some diagonal quadratic form of r variables

$$\dim T(\nu, Q) \leq \binom{\frac{\nu}{2} + r - 2}{r-2}.$$

In [4] the upper bounds for the dimensions of the spaces $T(\nu, Q)$ for some diagonal and nondiagonal quadratic forms are established, in a number of cases the dimension of the space $T(2, Q)$ is calculated and a basis of this space is constructed.

Gaigalas [5] obtained the upper bounds for the dimensions of the spaces $T(4, Q)$ and $T(6, Q)$ for some diagonal quadratic forms and presented the upper bounds of the dimensions of the spaces $T(\nu, Q)$ for some diagonal quadratic forms of six variables.

In this paper the dimension of the space $T(4, Q)$ is calculated for some diagonal quadratic forms of five variables and a basis of this space is constructed.

1 The spherical polynomials of the Space $\mathcal{P}(4, Q)$. Consider the diagonal quadratic form of five variables

$$Q(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}(x_4^2 + x_5^2),$$

where $0 < b_{11} < b_{22} < b_{33} < b_{44} = b_{55}$.

For the quadratic form $Q(X)$ we have

$$|A| = \det A = 2^5 b_{11}b_{22}b_{33}{b_{44}}^2, \qquad A_{11} = 2^4 b_{22}b_{33}{b_{44}}^2, \qquad A_{22} = 2^4 b_{11}b_{33}{b_{44}}^2, \\ A_{33} = 2^4 b_{11}b_{22}{b_{44}}^2, \quad A_{44} = A_{55} = 2^4 b_{11}b_{22}b_{33}b_{44}, \qquad A_{ij} = 0 \text{ for } \; i \neq j.$$

The total number of linearly independent spherical polynomials (2) of forth order for the quadratic forms of five variables is $\frac{1}{24}r(r^2-1)(r+6) = 55$, of which we need only the following:

$$\varphi_{1111}=x_1^4-\frac{2}{3b_{11}}Qx_1^2+\frac{1}{21b_{11}^2}Q^2,\qquad \varphi_{2222}=x_2^4-\frac{2}{3b_{22}}Qx_2^2+\frac{1}{21b_{22}^2}Q^2,$$
$$\varphi_{3333}=x_3^4-\frac{2}{3b_{33}}Qx_3^2+\frac{1}{21b_{33}^2}Q^2,\qquad \varphi_{4444}=x_4^4-\frac{2}{3b_{44}}Qx_4^2+\frac{1}{21b_{44}^2}Q^2,$$

$$\varphi_{1122}=x_1^2x_2^2-\frac{1}{9b_{11}}Qx_1^2-\frac{1}{9b_{22}}Qx_2^2+\frac{1}{126b_{11}b_{22}}Q^2,$$
$$\varphi_{1133}=x_1^2x_3^2-\frac{1}{9b_{11}}Qx_1^2-\frac{1}{9b_{33}}Qx_3^2+\frac{1}{126b_{11}b_{33}}Q^2,$$
$$\varphi_{3344}=x_3^2x_4^2-\frac{1}{9b_{33}}Qx_3^2-\frac{1}{9b_{44}}Qx_4^2+\frac{1}{126b_{33}b_{44}}Q^2.$$

2 The basis of the space $T(4,Q)$. For the constructed spherical polynomials, let's compile the corresponding generalized forth order theta-series (1).

$$\vartheta(\tau,\varphi_{1111},Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\varphi_{1111}\Big)z^n=\sum_{n=1}^{\infty}\sum_{Q(x)=n}\Big(x_1^4-\frac{2}{3b_{11}}Qx_1^2+\frac{1}{21b_{11}^2}Q^2\Big)z^n$$

$$=\frac{16}{21}z^{b_{11}}+\dots+\frac{2b_{22}^2}{21b_{11}^2}z^{b_{22}}+\dots+\frac{2b_{33}^2}{21b_{11}^2}z^{b_{33}}+\dots+\frac{4b_{44}^2}{21b_{11}^2}z^{b_{44}}+\dots$$

$$\vartheta(\tau,\varphi_{2222},Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\varphi_{2222}\Big)z^n=\sum_{n=1}^{\infty}\sum_{Q(x)=n}\Big(x_2^4-\frac{2}{3b_{22}}Qx_2^2+\frac{1}{21b_{22}^2}Q^2\Big)z^n$$

$$=\frac{2b_{11}^2}{21b_{22}^2}z^{b_{11}}+\dots+\frac{16}{21}z^{b_{22}}+\dots+\frac{2b_{33}^2}{21b_{22}^2}z^{b_{33}}+\dots+\frac{4b_{44}^2}{21b_{22}^2}z^{b_{44}}+\dots$$

$$\vartheta(\tau,\varphi_{3333},Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\varphi_{3333}\Big)z^n=\sum_{n=1}^{\infty}\sum_{Q(x)=n}\Big(x_3^4-\frac{2}{3b_{33}}Qx_3^2+\frac{1}{21b_{33}^2}Q^2\Big)z^n$$

$$=\frac{2b_{11}^2}{21b_{33}^2}z^{b_{11}}+\dots+\frac{2b_{22}^2}{21b_{33}^2}z^{b_{22}}+\dots+\frac{16}{21}z^{b_{33}}++\dots+\frac{4b_{44}^2}{21b_{33}^2}z^{b_{44}}+\dots$$

$$\vartheta(\tau,\varphi_{4444},Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\varphi_{4444}\Big)z^n=\sum_{n=1}^{\infty}\sum_{Q(x)=n}\Big(x_4^4-\frac{2}{3b_{44}}Qx_4^2+\frac{1}{21b_{44}^2}Q^2\Big)z^n$$

$$=\frac{2b_{11}^2}{21b_{44}^2}z^{b_{11}}+\dots+\frac{2b_{22}^2}{21b_{44}^2}z^{b_{22}}+\dots+\frac{2b_{33}^2}{21b_{44}^2}z^{b_{33}}+\dots+\frac{18}{21}z^{b_{44}}+\dots,$$

$$\vartheta(\tau,\varphi_{1122},Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\varphi_{1122}\Big)z^n=\sum_{n=1}^{\infty}\sum_{Q(x)=n}\Big(x_1^2x_2^2-\frac{1}{9b_{11}}Qx_1^2-\frac{1}{9b_{22}}Qx_2^2+\frac{1}{126b_{11}b_{22}}Q^2\Big)z^n$$

$$=(-\frac{2}{9}+\frac{b_{11}}{63b_{22}})z^{b_{11}}+\ldots+(-\frac{2}{9}+\frac{b_{22}}{63b_{11}})z^{b_{22}}+\dots+\frac{b_{33}^2}{63b_{11}b_{22}}z^{b_{33}}+\dots+\frac{2b_{44}^2}{63b_{11}b_{22}}z^{b_{44}}+\dots,$$

$$\vartheta(\tau,\varphi_{1133},Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\varphi_{1133}\Big)z^n=\sum_{n=1}^{\infty}\sum_{Q(x)=n}\Big(x_1^2x_3^2-\frac{1}{9b_{11}}Qx_1^2-\frac{1}{9b_{33}}Qx_3^2+\frac{1}{126b_{11}b_{33}}Q^2\Big)z^n$$

$$=(-\frac{2}{9}+\frac{b_{11}}{63b_{33}})z^{b_{11}}+\dots+\frac{b_{22}^2}{63b_{11}b_{33}}z^{b_{22}}+\dots(-\frac{2}{9}+\frac{b_{33}}{63b_{11}})z^{b_{33}}+\dots++\frac{2b_{44}^2}{63b_{11}b_{33}}z^{b_{44}}+\dots,$$

$$\vartheta(\tau,\varphi_{3344},Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\varphi_{3344}\Big)z^n=\sum_{n=1}^{\infty}\sum_{Q(x)=n}\Big(x_3^2x_4^2-\frac{1}{9b_{33}}Qx_3^2-\frac{1}{9b_{44}}Qx_4^2+\frac{1}{126b_{33}b_{44}}Q^2\Big)z^n$$

$$=\frac{b_{11}^2}{63b_{33}b_{44}}z^{b_{11}}+\dots+\frac{b_{22}^2}{63b_{33}b_{44}}z^{b_{22}}+\dots+(-\frac{2}{9}+\frac{b_{33}}{63b_{44}})z^{b_{33}}+\dots+(-\frac{2}{9}+\frac{2b_{44}}{63b_{33}})z^{b_{44}}+\dots.$$

These generalized theta-series are linearly independent since the determinant constructed from the coefficients of these theta-series is not equal to zero. In [4] we have

$$\dim T(\nu,Q)\leq\begin{cases}\frac{1}{6}\Big(\frac{\nu}{4}+1\Big)\Big(\frac{\nu}{4}+2\Big)(\nu+3) & \text{if } \nu\equiv 0(\text{mod}4),\\ \frac{1}{24}\Big(\frac{\nu}{2}+1\Big)\Big(\frac{\nu}{2}+3\Big)(\nu+7) & \text{if } \nu\equiv 2(\text{mod}4)\end{cases}$$

For $\nu=4$ we have $\dim T(4,Q)\leq 7$. Hence these theta-series form the basis of the space $T(4,Q)$. We have the following

Theorem 1. *Let $Q(X)$ be the diagonal quadratic form of five variables, given by $Q(X)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+b_{44}(x_4^2+x_5^2)$, then the generalized theta-series:*

$$\vartheta(\tau,\varphi_{1111},Q),\vartheta(\tau,\varphi_{2222},Q),vartheta(\tau,\varphi_{3333},Q),\vartheta(\tau,\varphi_{4444},Q),\vartheta(\tau,\varphi_{1122},Q),$$

$$\vartheta(\tau,\varphi_{1133},Q),\vartheta(\tau,\varphi_{3344},Q)$$

form the basis of the space $T(4,Q)$.

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