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ON THE APPROXIMATE SOLUTION OF THE J. BALL'S BEAM EQUATION IN THE CASE OF TEMPERATURE DEPENDENCE OF EFFECTIVE VISCOSITY

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Abstract. An initial-boundary value problem is posed for the J. Ball integro-differential equation, which describes the dynamic state of a beam. The solution is approximated utilizing the Galerkin method, stable symmetrical difference scheme and the Jacobi iteration method. This paper presents the approximate solution to one practical problem. Particularly, the results of numerical computations of the initial-boundary value problem for an iron beam. In the presented article the case where the effective viscosity depends on the temperature is discussed. The results of numerical calculations qualitatively satisfactorily describe the process under consideration.

1 Statement of the problem.

Let us consider the nonlinear equation

$$\begin{gathered} u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t) \\ -\left(\beta+\kappa\int_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t)-\sigma\left(\int_0^L u_x(x,t)\,u_{xt}(x,t)\,dx\right) \\ \times u_{xx}(x,t)=f(x,t), \quad 0<x<L, \quad 0<t\le T, \end{gathered} \tag{1}$$

with the initial boundary conditions

$$u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), \tag{2}$$

$$u(0,t)=u(L,t)=0, \quad u_{xx}(0,t)=u_{xx}(L,t)=0. \tag{3}$$

Here $\alpha,\gamma,\kappa,\sigma,\beta$, and δ are given constants, among which the first four are positive numbers, while $u^0(x)\in W_2^2(0,L)$ and $u^1(x)\in L_2(0,L)$, are given functions such that $u^0(0)=u^1(0)=u^0(L)=u^1(L)=0$.

The right-hand side function $f(x,t)\in L_2((0,L)\times(0,T))$. We suppose that there exits a solution $u(x,t)\in W_2^2((0,L)\times(0,T))$ of problem (1)-(3).

The presented article is a direct continuation of the articles [1]-[5] that consider the construction of algorithms and their corresponding numerical computations for the approximate solution of nonlinear integro-differential equations of the Timoshenko type. In particular, in this work, an initial-boundary value problem is considered for the J. Ball integro-differential equation, which describes the dynamic state of a beam (see, [6]). The solution is approximated utilizing the Galerkin method, stable symmetrical difference scheme and the Jacobi iteration method. In the articles [2]-[3] the algorithm is approved by tests. The articles [4]-[5] and the present paper consider the approximate solution to one practical problem, particularly, the results of numerical computations of the initial-boundary value problem for an iron beam are represented in the tables.

A physical model that J. Ball uses in the article [6] is taken from the handbook of Engineering Mechanics written by E. Mettler (see [7]). For this model he wrote the corresponding initial-boundary value problem for the integro-differential equation of beam (1) Here $\alpha, \gamma, \kappa, \sigma, \beta$ and δ are given constants from which the following five have the form

$$\alpha = \frac{E \cdot I}{\rho}, \quad \beta = \frac{E \cdot A \cdot \Delta}{L \cdot \rho}, \quad \gamma = \frac{\eta \cdot I}{\rho}, \quad \kappa = \frac{E \cdot A}{2L \cdot \rho}, \quad \sigma = \frac{A\eta}{L \cdot \rho}.$$

Here, E denotes Young's modulus, A is the cross-sectional area, η is the effective viscosity, I is the cross-sectional second moment of area, ρ is the mass per unit length in the reference configuration, L is beam length, Δ is beam length change (extension) and δ the coefficient of external damping.

2 The numerical realization. For the approximate solution of initial-boundary value problem (1)-(3) several programs are composed in Maple, several numerical experiments are carried out. This paper presents the approximate solution to one practical problem. Particularly, the results of numerical computations of the initial-boundary value problem for an iron beam are represented in the tables.

Issues of the initial-boundary value problem of the iron beam are studied for the following meanings of parameters: $L = 1\,\mathrm{m}$, time $T = 1\,\mathrm{sec}$, the grid length of a spatial variable $H = 20$, the grid length of a temporal variable $M = 20$, the amount of coordinate functions in the Galerkin method $n = 5$; number of iterations $n_{iter} = 5$,

$E = 1.9 \times 10^6 \,\frac{\mathrm{kg}}{\mathrm{cm}^2}$, $\rho = 7.874 \,\frac{\mathrm{g}}{\mathrm{cm}^3}$, $\Delta = 0.01\,\mathrm{m}$, $A = 0.01\,\mathrm{m}^2$, $I = 1000\,\mathrm{Pa}$.

We will see the case where the dependence of the effective viscosity on the temperature is discussed. The effective viscosity has the form

$\eta(t_m) = \dfrac{0.01775}{1 + 0.0337 * t_m + 0.000221 * t_m^2}$, with t_m - temperature C^0 (see worcs [8]-[9]); $t_m[i] = t_m[0] + dt_m * t[i]$; $t[i] = i * \tau; i = \overline{0, M}$; time $t \in [0, 1]$; $\alpha = 0.24613 \times 10^6 \cdot I$, $\beta = 241.3$, $\gamma = 0.12954 \times I \cdot \eta$, $\kappa = 12065$, $\sigma = 0.0127 \times \eta$, and $\delta = 0$. The initial functions $u^0(x) = \sin\left(\frac{\pi x}{L}\right)$, $u^1(x) = 0$, the right-hand function $f(x,t) \equiv 0$.

As the temperature increases, the viscosity decreases and therefore, naturally, the bends $u(x,t)$ increase with increasing modulus, which is confirmed by numerical experiments, see table 1. We considered basically two different cases: a). A simple model $dt_m = 0\ C^\circ$ (Case 1) - for each specific t we calculate η and in the corresponding difference equations we obtain constant coefficients γ, σ for all time layers; b). Complex model $dt_m \neq 0C^\circ$ - in difference equations we obtain coefficients γ, σ depending on t for all time layers; We will see two options:

1. non-extreme situation (Case 2-4: $dt_m = 0.01C^\circ; 0.1C^\circ; 1C^\circ$ (see Table 1);
2. extreme situation Case 5-7: $dt_m = 10C^\circ; 30C^\circ; 50C^\circ$ (see Table 1);

$\backslash x$ $t = \backslash$	$x =$ $0; 1$	Case	$x = 0.2$ $x = 0.8$	$x = 0.4$ $x = 0.6$	$Case$	$x = 0.2$ $x = 0.8$	$x = 0.4$ $x = 0.6$
0	0	1	0.58778525	0.95105652	1	0.58778525	0.9510565
0	0	2	0.58778525	0.95105652	5	0.58778525	0.95105652
0	0	3	0.58778525	0.95105652	6	0.58778525	0.95105652
0	0	4	0.58778525	0.95105652	7	0.58778525	0.95105652
0.25	0	1	5.29005757	8.55949295	1	5.29005757	8.55949295
0.25	0	2	5.29005757	8.55949295	5	5.29005757	8.55949315
0.25	0	3	5.29005757	8.55949295	6	5.29005757	8.55949352
0.25	0	4	5.29005759	8.55949298	7	5.29005812	8.55949384
0.5	0	1	11.1678526	18.0699615	1	11.1678526	18.0699651
0.5	0	2	11.1678526	18.0699651	5	11.1678536	18.0699667
0.5	0	3	11.1678527	18.0699652	6	11.1678567	18.0699695
0.5	0	4	11.1678528	18.0699653	7	11.1678576	18.0699718
0.75	0	1	17.0455607	27.5802966	1	17.0455607	27.5802966
0.75	0	2	17.0455607	27.5802966	5	17.0455640	27.5802966
0.75	0	3	17.0455609	27.5802969	6	17.0455692	27.5803019
0.75	0	4	17.0455611	27.5802972	7	17.0455734	27.5803171
1	0	1	22.9231426	37.0904239	1	22.9231426	37.0904239
1	0	2	22.9231426	37.0904239	5	22.9231502	37.0904361
1	0	3	22.9231432	37.0904248	6	22.9231616	37.0904547
1	0	4	22.9231436	37.0904254	7	22.9231702	37.0904686

Table 1. (non-extreme situation: Case 2-4, extreme situation: Case 5-7)

Conclusion. In the presented article the case where the effective viscosity depends on the temperature is discussed. As the temperature increases, the effective viscosity decreases, so the deflections increase. The results of numerical calculations qualitatively satisfactorily describe the process under consideration.

REFERENCES

1. Papukashvili, G., Peradze, J., Tsiklauri, Z. On a stage of a numerical algorithm for Timoshenko type nonlinear equation. *Proc. A.Razmadze Math.Inst., Tbilisi* **158** (2012), 67-77.

2. Papukashvili, A., Papukashvili, G., Sharikadze, M. Numerical calculations of the J. Ball nonlinear dynamic beam. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math., Tbilisi* **32** (2018), 47-50.

3. Papukashvili, A., Papukashvili, G., Sharikadze, M. On a numerical realization for a Timoshenko type nonlinear beam equation. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math., Tbilisi* **33** (2019), 51-54.

4. Papukashvili, A., Geladze, G., Vashakidze, Z., Sharikadze, M. On the Algorithm of an Approximate Solution and Numerical Computations for J. Ball Nonlinear Integro-Differential Equation. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math., Tbilisi* **36** (2022), 75-78.

5. Papukashvili, A., Geladze, G., Vashakidze, Z., Sharikadze, M. Numerical solution for J.Ball's beam equation with velocity dependent Effective viscosity. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math., Tbilisi*, **37** (2023), 35-38.

6. Ball, J.M. Stability theory for an extensible beam. *J. Differential Equations*, **41** (1973), 399-418.

7. Mettler, E. Dynamic buckling, in Handbook of Engineering Mechanics (S. Flugge, Ed.). *Chapter 62, Mc Graw-Hill, New York, 1962.*

8. Li, Y., Sun, X., Zhang, X. Study of the energy consumption of the piped vekicle hydraulic transportation. *Advances in Mechanical Enginering.* **11**, 11 (2019), Doi:10.1177/1687814019885541.

9. Chenyu Fang, Xuncheng Pan, Bo Zhang, Kuandi Zhang, Xiangru Lv. An Experimental Study on the Evolution Law of Roll Wave Parameters in Overland Flow. *Hydrological Processes.* **35**, 11 (2021), Doi:10.1002/hyp.14408.

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