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CONSTRUCTIVE STOCHASTIC INTEGRAL REPRESENTATION OF SOME PATH-DEPENDENT BROWNIAN FUNCTIONAL ⋆

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Abstract. Here we consider some path-dependent Brownian functionals and derive constructive formulas for the stochastic integral representation.

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1 Introduction. The question of representing Brownian functionals as a stochastic Ito integral with an explicit form of the integrand is investigated. In general, the representation of one martingale as a stochastic integral with respect to another martingale, when the martingale is adapted with respect to the natural flow of the σ-algebras of the last martingale, is called a martingale representation.

On the other hand, it is known that the Ito stochastic integral as a process under certain conditions on the integrand is a martingale with respect to the natural flow of σ-algebras of Brownian motion. The question naturally arises whether the opposite is true: can any martingale with respect to the natural flow of σ-algebras of Brownian motion be represented as a stochastic Ito integral? The well-known theorem of Clark (1970) gives a positive answer to this question (which is a particular case of solving the martingale representation problem), although an implicit answer can also be found in Ito (1951).

Formally, the task looks like this: assume that $(X_t, \Im_t^X := \sigma\{X_s : 0 \leq s \leq t\}), t \in [0,T]$ and $(M_t, \Im_t^X), t \in [0,T]$ martingales are given.The question is, can M_t be represented as a stochastic integral with respect to martingale X_t? That is, is there a process $\varphi_t, t \in [0,T]$ adapted to the flow of σ-algebras $\Im_t^X$ such that $M_t = \int_0^t \varphi_s dX_s$? It turned out that we have a positive answer to this question (Clark, 1970) when X_t is a standard Wiener process [1]. But in general this is not so.

Consider two independent standard Wiener processes $w = (w_t, t \geq 0)$ and $v = (v_t, t \geq 0)$ (Kallianpur, 1987). Suppose that $X_t = \int_0^t w_s dv_s$ and $\Im_t^X := \sigma\{X_s : 0 \leq s \leq t\}$. Then $\langle X \rangle_t$ is $\Im_t^X$-measurable, and since $\langle X \rangle_t = \int_0^t w_s^2 ds$, from here it follows that w_t^2 is also $\Im_t^X$-measurable. Hence, process $M = (M_t, t \geq 0)$, where $M_t = w_t^2 - t$, is a square integrable martingale adapted with filtration $(\Im_t^X, t \geq 0)$, but it cannot be represented as an integral with respect to the martingale X_t.

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[1]The Wiener process and the Brownian motion are the same.

It should be noted that finding the explicit expression for φ_t is a very difficult problem. In this direction, one general result is known, called Clark-Ocone formula (1984), according to which $\varphi_t = E[D_t F|\Im_t]$, where D_t is the so called Malliavin stochastic derivative. Glonti and Purtukhia (2017) generalized the Clark-Ocone formula to the case when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable, and proposed a method for finding the integrand.

Here we consider some path-dependent Brownian functionals and derive constructive formulas for the stochastic integral representation. The considered class of functionals is not stochastically smooth, and both the well-known Clark-Ocone formula (1984) and its generalization, the Glonti-Purtukhia formula ([1]), are not applicable to them. Here we use the earlier Jaoshvili-Purtukhia ([2]) generalization of the Clark-Ocone formula, according to which the following result is valid.

Theorem 1. *Let f and its generalized derivative ∂f be square-integrable functions with the weight function $\exp\{-\frac{x^2}{2}\}$ and $\xi \in D_{1,2}$, then the following stochastic integral representation is valid:*

$$f(\xi) = Ef(\xi) + \int_0^T E[\partial f(\xi) D_t \xi | \Im_t^B] dB_t \quad (P-a.s.)$$

In addition, we needed two technical results about ordinary integrals, which are obviously moments of the standard normal distribution in the case of a complete space ($a = -\infty$), which can be easily verified using the method of mathematical induction.

Proposition 1. *For any real number a and non-negative integer $n \geq 0$ we have*

$$\frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x^{2n} \exp\{-\frac{x^2}{2}\} dx =$$

$$= \varphi(a) \sum_{k=1}^{n} \frac{(2n-1)!!}{(2k-1)!!} a^{2k-1} + (2n-1)!! \big(1 - \Phi(a)\big),$$

where Φ is the standard normal distribution function and φ is its density function, $(2n-1)!! := 1 \cdot 3 \cdots (2n-1)$, $(-1)!! := 1$.

Proposition 2. *For any real number a and natural n we have*

$$\frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x^{2n-1} \exp\{-\frac{x^2}{2}\} dx = \varphi(a) \sum_{k=0}^{n-1} \frac{(2n-2)!!}{(2k)!!} a^{2k},$$

where $(2n)!! := 2 \cdot 4 \cdots (2n)$, $0!! := 1$.

2 Main results.

For any non-negative integer n we denote

$$G(2n+1) := \Big(\int_0^T B_s ds \Big)^{2n+1}$$

and consider the following path-dependent Brownian functional:

$$F(2n+1) := [G(2n+1)]^+.$$

Theorem 2. *For any non-negative integer n the following stochastic integral representation is valid:*

$$F(2n+1) = EF(2n+1)$$
$$+(2n+1)\sum_{k=0}^{2n}\int_0^T (T-t)C_{2n}^k\sigma^k\eta^{2n-k}I_k(\sigma,\eta)\big|_{\eta=\int_0^t(T-s)dB_s}dB_t,$$

where

$$\sigma^2 := (T-t)^3/3, \quad I_{2k-1}(\sigma,\eta) = \varphi\Big(\frac{\eta}{\sigma}\Big)\sum_{i=0}^{k-1}\frac{(2k-2)!!}{(2i)!!}\Big(\frac{\eta}{\sigma}\Big)^{2i},$$

$$I_{2k}(\sigma,\eta) = (2k-1)!!\phi\Big(\frac{\eta}{\sigma}\Big) + \varphi\Big(\frac{\eta}{\sigma}\Big)\sum_{i=1}^{k}\frac{(2k-1)!!}{(2i-1)!!}\Big(-\frac{\eta}{\sigma}\Big)^{2i-1}.$$

Proof. It is not difficult to see that

$$\int_t^T (T-s)dB_s \sim N(0,(T-t)^3/3) = N(0,\sigma^2).$$

According to Theorem 1, we have

$$\varphi(t,\omega) = E\big[I_{\{(\int_0^T B_s ds)^{2n+1}>0\}}(2n+1)\Big(\int_0^T B_s ds\Big)^{2n}\cdot\int_0^T I_{[0,s]}(t)ds|\Im_t^B\big]$$

$$= (2n+1)(T-t)E\big[I_{\{\int_0^T (T-s)dB_s>0\}}\Big(\int_0^T (T-s)dB_s\Big)^{2n}|\Im_t^B\big] = (2n+1)(T-t)$$

$$\times E\big[I_{\{\int_t^T (T-s)dB_s+\int_0^t (T-s)dB_s>0\}}\Big(\int_t^T (T-s)dB_s + \int_0^t (T-s)dB_s\Big)^{2n}|\Im_t^B\big].$$

Further, thanks to the Markov property of Brownian Motion, using the Newton binomial formula and the values of the moments of the normal distribution, it is not difficult to establish that

$$\varphi(t,y) = (2n+1)(T-t)E\big[I_{\{\int_t^T (T-s)dB_s>-y\}}\Big(\int_t^T (T-s)dB_s + y\Big)^{2n}\big]\big|_{y=\int_0^t (T-s)dB_s}$$

$$= \frac{(2n+1)(T-t)}{\sqrt{2\pi}\sigma}\sum_{r=0}^{2n}C_{2n}^r y^{2n-r}\int_{-y}^{+\infty}x^r\exp\{-\frac{x^2}{2\sigma^2}\}dx\big|_{y=\int_0^t (T-s)dB_s}$$

$$= (2n+1)(T-t)\sum_{r=0}^{2n}C_{2n}^r\sigma^r y^{2n-r}I_{2n}(r,\sigma,y)\big|_{y=\int_0^t (T-s)dB_s},$$

where

$$I_{2n}(r,\sigma,y) := \frac{1}{\sqrt{2\pi}}\int_{-y/\sigma}^{+\infty} x^r \exp\{-\frac{x^2}{2}\}dx.$$

On the other hand, from Proposition 1 and 2, for any natural r we have

$$I_{2n}(2r-1,\sigma,y) := \frac{1}{\sqrt{2\pi}}\int_{-y/\sigma}^{\infty} x^{2r-1} \exp\{-\frac{x^2}{2}\}dx$$

$$= \varphi(-\frac{y}{\sigma})\sum_{k=0}^{r-1}\frac{(2r-2)!!}{(2k)!!}(-\frac{y}{\sigma})^{2k} = \varphi(\frac{y}{\sigma})\sum_{k=0}^{r-1}\frac{(2r-2)!!}{(2k)!!}(\frac{y}{\sigma})^{2k}.$$

and

$$I_{2n}(2r,\sigma,y) := \frac{1}{\sqrt{2\pi}}\int_{-y/\sigma}^{\infty} x^{2r} \exp\{-\frac{x^2}{2}\}dx$$

$$= \varphi(-\frac{y}{\sigma})\sum_{k=1}^{r}\frac{(2r-1)!!}{(2k-1)!!}(-\frac{y}{\sigma})^{2k-1} + (2r-1)!!(1-\phi(-y/\sigma))$$

$$= \varphi(\frac{y}{\sigma})\sum_{k=1}^{r}\frac{(2r-1)!!}{(2k-1)!!}(-\frac{y}{\sigma})^{2k-1} + (2r-1)!!\phi(y/\sigma).$$

□

Corollary. The following stochastic integral representation is valid:

$$\Big(\int_0^T B_s ds\Big)^+ = \sqrt{\frac{T^3}{6\pi}} + \int_0^T (T-t)\phi\Big(\sigma^{-1}\int_0^t (T-s)dB_s\Big)dB_t.$$

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