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ONE PROBLEM FOR THE COUPLED LINEAR THEORY OF PLANE ELASTICITY FOR POROUS MATERIALS

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Abstract. In the present paper the linear coupled model of elastic porous materials is considered, which takes into account the coupled phenomenon of the concepts of Darcy's law and volume fraction. The general solution of the two-dimensional system of equations of plane deformation is represented by means of three analytic functions of a complex variable and solution of Helmholtz equation. The Dirichlet boundary value problem is solved for the circle.

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1 Introduction.

The theories of porous media have found applications in many branches of civil engineering, geotechnical engineering, technology, hydrology, and recent years, medicine and biology

The consolidation (quasi-static) theory of poroelasticity for isotropic porous materials on the basis of Darcy's law is presented in the pioneering work by Biot [1]. Nunziato and Cowin [2] introduced a theory for the behavior of deformable porous materials based on the volume fraction concept.

Many of the engineering problems have multiphysics nature and we encounter various coupled processes in porous media. Recently, Svanadze [3, 4] introduced the linear models of elasticity and thermoelasticity for single porosity materials in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction of pore network is considered. The basic boundary value problems of these models are studied by Bitsadze [5] and Tsagareli [6].

2 The basic equations.

Let (x_1, x_2, x_3) be a point of the Euclidean three dimensional space R^3. In what follows we consider an isotropic and homogeneous porous elastic solid occupies a region of R^3. The governing system of homogeneous equations of motion in the coupled linear theory for porous materials consists of the following sets of equations [3, 4]:

- Equations of motion

$$\partial_j t_{ji} = 0, \quad \partial_j \sigma_j + \xi = 0, \quad i, j = 1, 2, 3, \tag{1}$$

where t_{ij} is the component of the stress tensor, σ_i is the component of the equilibrated stress associated to the pore network, $\partial_j = \frac{\partial}{\partial x_j}$, the function ξ is the intrinsic equilibrated body force and is defined by

$$\xi = -be_{jj} - \alpha_1 \varphi + mp, \tag{2}$$

where φ is the change of volume fraction of pores, p is the change of fluid pressure in pore network, e_{ij} is the component of strain tensor and is defined by

$$e_{ij} = \tfrac{1}{2}\left(\partial_j u_i + \partial_i u_j\right), \tag{3}$$

where u_i are the components of the displacement vector in solid.

- Constitutive equations

$$\begin{aligned} &t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + (b\varphi - \beta p)\delta_{ij}, \\ &\sigma_i = \alpha' \partial_i \varphi, \end{aligned} \tag{4}$$

where λ and μ are the Lamé constants, β is the effective stress parameter, δ_{ij} is the Kronecker delta, the values b, m, α' and α_1 are the constitutive coefficients.

- From Equation of fluid mass conservation and Darcy's law

$$k\triangle p = 0, \quad k = \tfrac{k'}{\mu'}, \tag{5}$$

where μ' is the fluid viscosity, k' is the macroscopic intrinsic permeability associated with the pore network.

3 The plane deformation. In the case of plane deformation $u_3 = 0$ while the functions u_1, u_2, φ and p do not depend on the coordinate x_3 [7].

On the plane Ox_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\alpha}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar z} = 0.5(\partial_1 + i\partial_2)$, $\bar z = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar z}$.

From (1)-(5) we obtain the following system of equations of motion in the coupled linear theory for porous materials expressed in terms of the components of the displacement vector field u_1, u_2, the change of volume fraction φ, the change of pressure p (in the complex form)

$$\begin{aligned} &\mu\Delta u_+ + 2(\lambda+\mu)\partial_{\bar z}\theta + 2b\partial_{\bar z}\varphi - 2\beta\partial_{\bar z} p = 0, \\ &(\alpha'\Delta - \alpha_1)\varphi - b\theta + mp = 0, \\ &k\Delta p = 0. \end{aligned} \tag{6}$$

Theorem 1. *The general solution of system (6) is represented as follows [7, 8]:*

$$\begin{aligned} 2\mu u_+ &= \varkappa g(z) - z\overline{g'(z)} - \overline{h(z)} - k_1(f(z) + z\overline{f'(z)}) - k_2\partial_{\bar z}\chi(z,\bar z), \\ \varphi &= \chi(z,\bar z) - k_3(g'(z) + \overline{g'(z)}) + k_4(f'(z) + \overline{f'(z)}), \\ p &= f'(z) + \overline{f'(z)}, \end{aligned} \tag{7}$$

where $g(z)$, $h(z)$ and $f(z)$ are the arbitrary analytic functions of a complex variable z, $\chi(z,\bar z)$ is an arbitrary solution of the Helmholtz equation

$$\Delta\chi - \gamma^2\chi = 0, \quad \gamma^2 = \frac{1}{2}\left(\alpha_1 + \frac{b^2}{\lambda + 2\mu}\right),$$

and

$$\varkappa = \frac{\nu}{1-\nu}, \quad \nu = \frac{b(\lambda+\mu)}{\alpha_1(\lambda+2\mu)+b^2} - \frac{\lambda+\mu}{2(\lambda+2\mu)},$$

$$k_1 = \frac{2\lambda+3\mu}{\lambda+2\mu}\left(\beta + \frac{b^2\beta+bm(\lambda+\mu)}{b^2+\alpha'(\lambda+2\mu)}\right), \quad k_2 = \frac{4b(2\lambda+3\mu)}{\gamma^2(\lambda+2\mu)},$$

$$k_3 = \frac{b(\lambda+2\mu)}{(1-\nu)(\alpha_1(\lambda+2\mu)+b^2)}, \quad k_4 = \frac{b\beta+m(\lambda+\mu)}{b^2+\alpha'(\lambda+2\mu)}.$$

4 The Dirichlet problem for the circle. Let us consider the elastic circle bounded by the circumference of radius R (Fig. 1). The origin of coordinates is at the center of the circle [8].

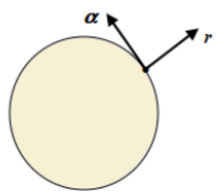

Figure 1: the poroelastic circle.

On the circumference, we consider the following boundary value problem

$$u_+ = A, \;\; \varphi = B, \;\; p = C, \quad r = R, \tag{8}$$

where A, B and C are sufficiently smooth functions.

The analytic functions $g(z)$, $h(z)$, $f(z)$ and the metaharmonic function $\chi(z,\bar{z})$ are represented as the series [7]

$$g(z) = \sum_{n=1}^{\infty} a_n z^n, \;\; h(z) = \sum_{n=0}^{\infty} b_n z^n, \;\; f(z) = \sum_{n=1}^{\infty} c_n z^n, \;\; \chi(z,\bar{z}) = \sum_{-\infty}^{+\infty} \alpha_n I_n(\gamma r) e^{in\vartheta}, \tag{9}$$

where $I_n(\cdot)$ are the modified Bessel functions of the first kind of n-th order.

Expand the function $A/2\mu \cdot e^{i\alpha}$, B and C, given on $r = R$, in a complex Fourier series

$$\frac{A}{2\mu} e^{i\alpha} = \sum_{-\infty}^{\infty} A_n e^{in\alpha}, \quad B = \sum_{-\infty}^{\infty} B_n e^{in\alpha}, \quad C = \sum_{-\infty}^{\infty} C_n e^{in\alpha}. \tag{10}$$

Substituting (7), (9), (10) into the boundary conditions (8) and comparing the coefficients of $e^{in\alpha}$ we have

$$\varkappa R a_1 - R\bar{a}_1 - k_1 R c_1 - k_1 R \bar{c}_1 - \frac{\gamma k_2}{2} I_1(\gamma R)\alpha_0 = A_1,$$

$$
\begin{aligned}
&-(n+2)R^{n+2}a_{n+2} - R^n b_n - k_1(n+2)R^{n+2}c_{n+2} - \frac{\gamma k_2}{2} I_n(\gamma R)\alpha_{n+1} \\
&= \overline{A}_{-n}, \quad n \geq 0 \\
&\varkappa R^n a_n - k_1 R^n c_n - \frac{\gamma k_2}{2} I_n(\gamma R)\alpha_{n-1} = \overline{A}_n, \quad n \geq 2 \\
&I_n(\gamma R)\alpha_n - k_3(n+1)R^{n+1}a_{n+1} + k_4(n+1)R^{n+1}c_{n+1} = B_n, \quad n \geq 0 \\
&nR^n c_n = C_{n-1}, \quad n \geq 1.
\end{aligned} \tag{11}
$$

From system (11) we can find all coefficients $a_n,\ b_n,\ c_n,\ \alpha_n$.

It is easy to prove the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

The procedure of solving a boundary value problem remains the same when stresses and change in volume fraction on the domain boundary are given arbitrarily, but the condition that the principal vector and the principal moment of external forces are equal to zero is fulfilled.

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