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SOME FUNCTIONAL PROPERTIES OF ELLIPTIC SYSTEMS *

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Abstract. This article presents some results of a study of the functional properties of generalized entire analytic functions. The generalized regular solution of the regular (with regular coefficients) system of Carleman-Bers-Vekua is called a generalized analytical function. The results, in a somewhat modified form, allow generalization for a sufficiently wide class of multidimensional elliptic systems as well.

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Everywhere below $\mathbb{R}$- is the real number axis, $\mathbb{C}_{z=x+iy}$ is the plane of complex numbers, and $\mathbb{A}$- is a class of all possible entire analytic functions (monogeneous functions at every point z of the complex plane $\mathbb{C}$) of a complex variable $z = x + iy$. One of the current directions of contemporary mathematical analysis is the systematic study of the mentioned class of functions; it originates from the second half of the 19th century in the classical works of Weierstrass, Mittag-Lefleur, Picard and others. One of the most important results of the theory of entire functions is the classical Liouville theorem, according to which every entire $f(z)$ function (without any exceptions) is either constant or unbounded. Moreover, if the rate of growth of the entire function $f(z)$ in the neighborhood of the point at infinity does not exceed the rate of growth of the function $|z|^N$, i.e.

$$f(z) = \mathcal{O}(|z|^N), z \to \infty, \tag{1}$$

for some non-negative integer N, then the function $f(z)$ is a polynomial, whose order does not exceed N.

There are a very few non-trivial regularities that every member of this class obeys. Therefore, it is quite natural to separate subclasses from it (on the basis of certain criteria) and to study these subclasses individually rather than to study the class $\mathbb{A}$ entirely. As for a such criterion, the characteristic parameters of the growth of the value of the entire function $f(z)$ when the modulus of its argument $|z| \to \infty$ can be used. To obtain these parameters, in some sense simple function $h(z)$ is fixed, which is unbounded in the neighborhood of the point at infinity and from the whole class of functions $\mathbb{A}$, the subclass is separated, each representative of which grows no faster than this fixed function $h(z)$. From this point of view, the function $h(z) = |z|^N$ with exponential growth by virtue of

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the classical theorem of Liouville is uninteresting. As it turned out, from the mentioned point of view, the function with exponential growth is very interesting and important

$$H(z) = \exp\left\{|z|^{\delta}\right\}.$$

Here, the positive number $\delta \in \mathbb{R}$, through which the most important (both theoretically and practically) subclasses are separated from the whole class $\mathbb{A}$ of the entire functions $f(z)$. These subclasses probably appeared for the first time in Hadamard's work [2]. In particular, the most important classification of all functions according to exponential growth in the neighborhood of the point at infinitely is given in the above mentioned work. Hadamars classification of all analytical functions is the basis of the most important directions of research in complex analysis. A similar classification of generalized analytical functions is given in this work.

Suppose that $W = W(z)$ is a continuous complex function of a complex variable whose area of definition is the whole complex plane $\mathbb{C}$. Let's use the classical scheme (see e.g. [2, 3]) and match this function with a non-negative real number $R(W)$ or symbol $+\infty$ as follows: if for any positive number $\delta \in \mathbb{R}$ there exists a set $G \subset \mathbb{C}$ such that

$$\sup\left\{|W(z)| \exp(-|z|^{\delta}) : z \in G\right\} = +\infty, \tag{2}$$

then

$$R(W) = +\infty. \tag{3}$$

In such a case, we will say that the function $W = W(z)$ has an infinite exponential rating in the neighborhood of the point at infinity (in short, it has an infinite rating). If (2)-(3) does not take place, then it is obvious that there exists a non-negative number $\delta \in \mathbb{R}$ for the function $W = W(z)$ for which the condition is fulfilled

$$W(z) = \mathcal{O}(\exp\{|z|^{\delta}\}), z \to \infty. \tag{4}$$

For the function $W = W(z)$ with the finite rating, the number δ satisfying condition (4) is not uniquely defined; for every such number δ, condition (4) is satisfied by any number greater than this number (and possibly some are less than this number). Let us denote the set of all such possible numbers by Δ_W and let us call a non-negative real number

$$\varrho = \inf \Delta_W$$

an exponential rating of a function in the neighborhood of the point at infinity (shortly the rating). The following system of elliptic equations, whose complex form is

$$\frac{\partial w}{\partial \bar{z}} + aw + b\bar{w} = 0, \tag{5}$$

is called the Carleman-Bers-Vekua equation.

Everywhere below we mean that the coefficients of this equation satisfy the regularity condition, that is, they belong to the special class $L_{p,2}(\mathbb{C})$, for the number $p > 2$. The solution of this equation is understood as a continuous generalized solution over the whole complex plane $\mathbb{C}$ (see [1, 4]).

Let us denote the class of all possible solutions of the system (5) by $\mathbb{A}(a,b)$. Let $\mathbb{A}_{\varrho}(a,b)$ denote all elements of this class whose rating is ϱ.

The class of generalized analytic functions $\mathbb{A}(a,b)$ can be represented as a disjunctive union

$$\mathbb{A}(a,b) = \bigsqcup_{0\le\varrho\le+\infty} A_{\varrho}(a,b);$$

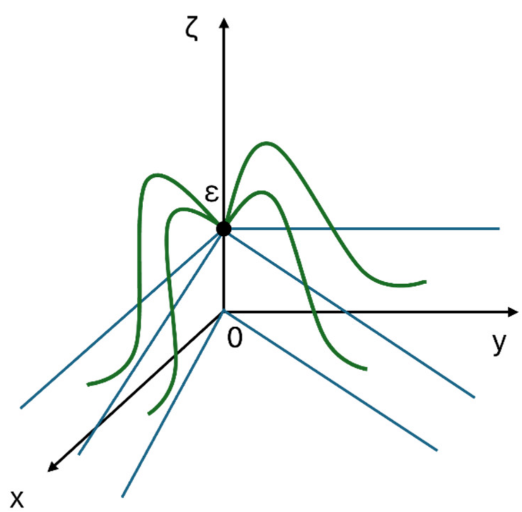
furthermore, for every non-negative number $\varrho \in \mathbb{R}$

$$card\ \mathbb{A}_{\varrho}(a,b) = card \mathbb{A}_{\infty}(a,b) = card c.$$

Theorem 1. *For every point z_0 of the plane $\mathbb{C}$, there exists $W_* \in \mathbb{A}(a,b)$ such that for every number $\varphi \in \mathbb{R}$,*

$$\lim_{r\to+\infty} W_*\left(z_0 + re^{i\varphi}\right) = 0$$

The exotic nature of the entire generalized analytic function $W_*(z)$ mentioned in Theorem 1 is partially reflected in the figure given by a fragment of the graph of a function $\varsigma = |W_*(z)|$ in the space $\mathbb{R}^3_{x,y,\varsigma}$.



We should take into account the most important fact that the function $\varsigma = |W_*(z)|$ is unbounded. The function $W_*(z)$ of the mentioned type "lives" only in the class $\mathbb{A}_{\infty}(a,b)$; For any number $\varrho \in [0,+\infty)$ in the class of functions $\mathbb{A}_{\varrho}(a,b)$ there is no function of type $W_*(z)$.

Theorem 2. *For every real number $\varrho > 0$, for every point z_0 of the plane $\mathbb{C}$, for every generalized entire analytic function $W \in \mathbb{A}_{\varrho}(a, b)$, there exists a number $\varphi \in \mathbb{R}$ such that*

$$\sup\left\{\left|W(z_0 + re^{i\varphi})\right|,\ r \geq 0\right\} = +\infty \tag{6}$$

Conclusion (6) of Theorem 2 is valid even when the number $\varrho = 0$, but the case of trivial W function should be excluded.

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