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THE MINIMAL REVERSE ENTROPY MARTINGALE MEASURE IN THE TRINOMIAL FINANCIAL MODEL

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Abstract. In this paper we consider incomplete financial market, where evolution of risky asset is described by trinomial scheme and construct the martingale measure which minimizes the reverse relative entropy.

Keywords and phrases: Financial market, trinomial model, martingale measure, reverse relative entropy.

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1 Content. Let us consider a real valued process $S = (S_n, \mathcal{F}_n), n = 0, 1, 2, ..., N$, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$, as an evolution of risky asset price on financial market, such that

$$S_n = S_{n-1}(1 + \rho_n), \tag{1}$$

where $S_0 > 0$ is a constant, $(\rho_n)_{n\geq 1}$, is the sequence of independent identically distributed random variables that take three values a, b, c with the probabilities p, q, l respectively,$p+q+l = 1$. We assume that $a < b < c$ and $-1 < a < 0 < c$. This model is known as a trinomial scheme. Here the reference measure P is defined by p, q, l on $\Omega = \{a, b, c\}^N$.

The measure Q is a martingale measure for S if Q is equivalent to P and $S = (S_n, \mathcal{F}_n)$ is a martingale with respect to Q. The martingale condition

$$E_Q[\Delta S_n/\mathcal{F}_{n-1}] = 0$$

implies that

$$a\tilde{p} + b\tilde{q} + c\tilde{l} = 0 \tag{2}$$

and the class of martingale measures $M(P)$ for S is defined by $\tilde{p}, \tilde{q}, \tilde{l}$, which satisfy the condition (2).

It can be shown, that density $\frac{dQ}{dP} = Z_N(\rho_1, \rho_2, ..., \rho_N), Q \in M(P)$ has the following form ($I(x)$ is the indicator of x):

$$\begin{aligned} Z_N = Z_N(\rho_1, \rho_2, ..., \rho_N) = \prod_{k=1}^{N} \left(\frac{\tilde{p}}{p} I(\rho_k = a) + \frac{\tilde{q}}{q} I(\rho_k = b) \right. \\ \left. + \frac{\tilde{l}}{l} I(\rho_k = c) \right) = \prod_{k=1}^{N} \xi_k, \end{aligned} \tag{3}$$

where

$$\xi_k = \frac{\tilde{p}}{p} I(\rho_k = a) + \frac{\tilde{q}}{q} I(\rho_k = b) + \frac{\tilde{l}}{l} I(\rho_k = c). \tag{4}$$

Definition 1. The reverse relative entropy $RE(Q,P)$ ([1]) of the probability measure Q with respect to probability measure P is defined as

$$RE(Q,P) = \begin{cases} E_P\left[-\ln \dfrac{dQ}{dP}\right], \ if \ \ Q << P \\ \qquad +\infty, \quad otherwise. \end{cases}$$

Definition 2. The reverse relative entropy minimal martingale measure ([1], [2]) is the measure Q^* for which

$$RE(Q^*,P) = \min_{Q \in M(P)} RE(Q,P).$$

For our trinomial scheme (1) and from (3), (4) we get

$$\begin{aligned} RE(Q,P) &= E\left[-\ln \frac{dQ}{dP}\right] = E\left[-\ln \prod_{k=1}^{N} \xi_k] = -\sum_{k=1}^{N} E[\ln \xi_k\right] \\ &= -NE[\ln \xi_1] = -N\left[p \ln \tfrac{\tilde{p}}{p} + q \ln \tfrac{\tilde{q}}{q} + l \ln \tfrac{\tilde{l}}{l}\right]. \end{aligned} \tag{5}$$

Now we consider the minimization problem of $RE(Q,P)$ given by (5) over all martingale measures $Q \in M(P)$ using Lagrange multiplier method ([3]). The Lagrangian has the following form

$$\begin{aligned} \Phi(\tilde{p},\tilde{q},\tilde{l}) = -&\left[p \ln \frac{\tilde{p}}{p} + q \ln \frac{\tilde{q}}{q} + l \ln \frac{\tilde{l}}{l}\right] \\ +\lambda &\left[a\tilde{p} + b\tilde{q} + c\tilde{l}\right] + \mu\left[\tilde{p} + \tilde{q} + \tilde{l} - 1\right] \end{aligned}$$

and from optimality conditions

$$\frac{\partial \Phi(\tilde{p},\tilde{q},\tilde{l})}{\partial \tilde{p}} = 0, \quad \frac{\partial \Phi(\tilde{p},\tilde{q},\tilde{l})}{\partial \tilde{q}} = 0, \frac{\partial \Phi(\tilde{p},\tilde{q},\tilde{l})}{\partial \tilde{l}} = 0,$$

we obtain

$$-\frac{\tilde{p}}{p} + \lambda a + \mu = 0, \ -\frac{\tilde{q}}{q} + \lambda b + \mu = 0, \ -\frac{\tilde{l}}{l} + \lambda c + \mu = 0.$$

Using this equalities and also taking into account (2) we can determine constants $\tilde{p}, \tilde{q}, \tilde{l}, \lambda, \mu$. Namely we get

$$\tilde{p} = \frac{p}{\lambda a + 1}, \ \tilde{q} = \frac{q}{\lambda b + 1}, \ \tilde{l} = \frac{l}{\lambda c + 1}, \ \ \mu = 1.$$

and λ satisfies the equation

$$\lambda^2 abc + \lambda[pa(b+c) + qb(a+c) + lc(a+b)] + ap + bq + cl = 0. \tag{6}$$

Thus we proved the following

Theorem 1. *The minimal reverse relative entropy martingale measure Q in the trinomial model (1) is determined by the probabilities*

$$\tilde{p} = \frac{p}{\lambda a + 1}, \ \tilde{q} = \frac{q}{\lambda b + 1}, \ \tilde{l} = \frac{l}{\lambda c + 1},$$

where λ satisfy equation (6).

Corollary 1. In the particular symmetrical case, when $a = -\alpha, b = 0, c = \alpha$, with $(\alpha > 0)$, equation (6) has the solution $\lambda = \dfrac{l-p}{\alpha(l+p)}$ and we obtain

$$\tilde{p} = \frac{p+l}{2}, \ \tilde{q} = q, \ \tilde{l} = \frac{p+l}{2}.$$

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