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ON ONE CLASS OF SOLUTIONS OF 2D NAVIER-STOKES EQUATIONS FOR THE INCOMPRESSIBLE FLUIDS

Nino Khatiashvili

Abstract. We consider 2D incompressible steady fluid flow in the finite and infinite areas. The velocity components of the flow satisfy the nonlinear Navier - Stokes equations (NSE) with the suitable boundary conditions. We modify NSE and find new class of solutions.The novel exact solutions of NSE are obtained in some specific cases.

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In space R_2 let us consider the area D with the boundary S. D is the finite or infinite domain of plane xOy. In the paper we study the steady incompressible Newtonian fluid flow in D. The governing system of equations is the stationary Navier-Stokes equations (NSE) with the equation of continuity [1-7]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+F_x+\nu\Delta u, \tag{1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+F_y+\nu\Delta v, \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3}$$

where $\vec{V}(u,v)$ is the velocity, $\vec{F}(F_x,F_y)$ is the body force, P is the pressure, ρ is the density, ν is the viscosity.

The system (1), (2), (3) is considered with the boundary conditions

$$u(x,y)|_S=0,\ v(x,y)|_S=0, \tag{4}$$

In [6] the system (1),(2),(3) was reduced to the system

$$-2\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)^2+4\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}-2\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)^2+F^*\frac{\partial^2 u}{\partial y^2}-\frac{\partial F^*}{\partial y}\frac{\partial u}{\partial y}=0, \tag{5}$$

$$-2\frac{\partial^2 v}{\partial y^2}\left(\frac{\partial v}{\partial x}\right)^2+4\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y\partial x}-2\frac{\partial^2 v}{\partial x^2}\left(\frac{\partial v}{\partial y}\right)^2+F^*\frac{\partial^2 v}{\partial x^2}-\frac{\partial F^*}{\partial x}\frac{\partial v}{\partial x}=0, \tag{6}$$

where

$$F^* = -\frac{1}{\rho}\Delta P + div\vec{F}.$$

Let us introduce a new function ψ, which is double differentiable in D and satisfies the condition $\psi|_S = 0$.

Let us suppose

$$u = F_1(\psi). \tag{7}$$

By (7) the equation (5) can be rewritten in the form

$$\left(\frac{\partial F_1}{\partial \psi}\right)^3 [-2\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial \psi}{\partial y}\right)^2 + 4\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - 2\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial \psi}{\partial x}\right)^2 + F^*\frac{\partial^2 u}{\partial y^2} - \frac{\partial F^*}{\partial y}\frac{\partial u}{\partial y} = 0, \tag{8}$$

Let us suppose

$$-\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial \psi}{\partial y}\right)^2 + 2\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial \psi}{\partial x}\right)^2 = 0. \tag{9}$$

(For example, (9) is valid for $\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x}$.)

Taking into account (9) from (8) one obtains

$$F^*\frac{\partial^2 u}{\partial y^2} - \frac{\partial F^*}{\partial y}\frac{\partial u}{\partial y} = 0. \tag{10}$$

If $F^* \neq 0$ from (10) we get

$$u = f_2(x) + \int_{y_0}^{y} f_1(x)\, F_0^*(x,t)dt;\ F_0^*(x,y) = \frac{F^*(x,y)}{F^*(x,y_0)};\ (x,y_0) \in D, \tag{11}$$

where $f_1(x)$, $f_2(x)$ are some double differentiable functions.

After simple transformations from (5),(6),(7),(11) we obtain

$$\begin{aligned} u \;=\; & \frac{1}{F^*}\Big(f_2'(x) + \frac{\partial}{\partial x}\int_{y_0}^{y} f_1(x)F_0^*(x,t)dt\Big)\Big(F_1^* + 2\nu\frac{\partial}{\partial y}\big(f_1(x)F_0^*(x,y)\big) \\ & -2\nu f"_2(x) - 2\nu\frac{\partial^2}{\partial x^2}\int_{y_0}^{y} f_1(x)F_0^*(x,t)dt\Big) + \frac{\nu}{F^*}\frac{\partial F^*(x,y)}{\partial x} \\ & +\frac{2F_2^* f_1(x)F_0^*(x,y)}{F^*} + \frac{2\nu}{F^*}\frac{\partial}{\partial x}\big(f_1(x)F_0^*(x,y)\big)\Big(\frac{F^*(x,y_0)}{2f_1(x)} \\ & -\Big(f_2'(x) + \frac{\partial}{\partial x}\int_{y_0}^{y} f_1(x)F_0^*(x,t)dt\Big)^2\frac{1}{f_1(x)F_0^*(x,y)} - f_1(x)F_0^*(x,y)\Big), \end{aligned} \tag{12}$$

$$v \;=\; \Big(\frac{\partial F_1^*(x,y)}{\partial y} + \nu\frac{\partial^2 F_0^*(x,y)}{\partial y^2} + \frac{\nu}{f_1}\frac{\partial^2 f_1 F_0^*(x,y)}{\partial x^2} \tag{13}$$

$$-\frac{f_2}{f_1}\frac{\partial f_1 F_0^*(x,y)}{\partial x}-\frac{\partial f_1 F_0^*(x,y)}{\partial x}\int_{y_0}^{y} f_1(x)F^*(x,t)dt\Big):\frac{\partial F_0^*(x,y)}{\partial y},$$

where $F_1^* = -\frac{1}{\rho}\frac{\partial P}{\partial x}+F_x;\ F_2^* = -\frac{1}{\rho}\frac{\partial P}{\partial y}+F_y$, and the functions f_1, f_2 satisfy the following system

$$f_2 = \frac{2f_2'}{F^*}\Big(F_1^*(x,y_0)+2\nu f_1\frac{\partial}{\partial y}\big(F_0^*(x,y)\big)-2\nu f"_2\Big)+\frac{\nu}{F^*}\frac{\partial F^*(x,y_0)}{\partial x} \tag{14}$$

$$-\frac{2f_1F_2^*(x,y_0)}{F^*}-\frac{2\nu f_1'}{f_1\,F^*}\Big(\frac{F^*(x,y_0)}{2}-\big(f_2'\big)^2+f_1^2\Big),\ for\, y=y_0,$$

$$f_2' = \Big(f_1'-\frac{\partial^2 F_1^*(x,y)}{\partial y^2}-\nu\frac{\partial^3 F_0^*(x,y)}{\partial y^3}-\frac{\nu}{f_1}\frac{\partial^3 f_1F_0^*(x,y)}{\partial y\partial x^2} \tag{15}$$

$$+\frac{\nu f_2}{f_1}\frac{\partial^2 f_1F_0^*(x,y)}{\partial y\partial x}\Big):\frac{\partial F_0^*(x,y)}{\partial y}+\frac{\partial^2 F_0^*(x,y)}{\partial y^2}$$

$$\Big(\frac{\partial F_1^*(x,y)}{\partial y}+\nu\frac{\partial^2 F_0^*(x,y)}{\partial y^2}-\frac{\nu f"_1+2f_2'f_1}{f_1}\Big):\Big(\frac{\partial F_0^*(x,y)}{\partial y}\Big)^2,\ for\, y=y_0,$$

$$f_2' = \sqrt{\frac{F^*(x_0,y_0)}{2}},\ \ f_1(x_0,y_0)=0;\ (x_0,y_0)\in S.$$

Example. For the fluid flow in the whole plane $x0y$ the solutions of problem (1), (2), (3), (4) are

$\psi = exp\,(x-y),\ u = exp\,(x-y)-1,\ v = exp\,(x-y)-exp\,(-2x)-1-2\nu,\ P = \rho\, exp\,(x-y),\ F_x = 0,\ F_y = 2exp\,(x-y)+4\nu exp\,(-2x)-2exp\,(-2x).$

Conclusion. The solutions of problem (1), (2), (3), (4) are given by the formulas (12), (13), if the functions f_1, f_2 satisfy the system (14), (15).

Remark 1. If the condition (10) is not fulfilled, there exist the solutions of the problem (1),(2),(3),(4). For example,in the domain $x>0$ the solutions can be given by

$$u = Aexp(x)\,sin(y),\ v = Aexp(x)\,cos(y)-Aexp(x),\ P = -\frac{A^2\rho}{2}exp(2\,x)$$

$$F_x = -A^2\,exp^2(x)\,cos(y),\ \ F_y = A\nu\, exp(x),$$

where A is some constant.

Remark 2. The solutions of problem (1), (2), (3), (4) of different classes are given in [1-7].

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Author(s) address(es):

Nino Khatiashvili
I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University
University str. 11, 0186 Tbilisi, Georgia
E-mail: ninakhatia@gmail.com