Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 38, 2024

ABOUT THE REGULAR COULOMB WAVE FUNCTION *

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Abstract. The regular Fourier image of the two-particle Coulomb wave function is described. It is rigorously shown that the above function exists in the sense of the generalized functions. This function belongs to the set of the Hilbert space functions and accordingly represents the full set of quantum-mechanical functions of the continuous spectrum. It has the simple analytical structure and satisfies the two-particle homogeneous equation of the perturbation theory.

Keywords and phrases: Two body potential quantum scattering theory, special functions, generalized functions, Fourier transform.

AMS subject classification (2010): 81U05, 33E99, 30G99, 42B10.

1 Motivation. It is known, that in the few body scattering theory the two-particle off shell functions play an essential role ([1], pp. 267 - 269).

The functions mentioned above in the case of the Coulomb field due to the long-range nature of the Coulomb potential are incorrect on the half shell [2].

The mentioned reason greatly complicates the solvability of the integral equations of the several charged particles of the continuous spectrum.

Using the nonstationary Schrodinger formalism for the Coulomb field, Dollard showed that the renormalized two-particle Coulomb scattering operator exists, and it is correct in the Hilbert space ([3], pp. 27 30, see also [4], pp. 513 516). Let us note, that the more rigorous mathematical formulation of the above result was proposed by Volker Enns [5].

Despite the predictions mentioned above, the existence of the Coulomb functions in the momentum space was questionable for some time. Somewhat later, the opinion appeared that the above quantum-mechanical functions in the momentum space can be considered in the sense of the generalized functions ([6], [7], pp. 41 51 and 63 - 67).

The fact mentioned above, as well as the studies in the complex analysis became the reason for our interest in these issues.

While studying these issues, we took into account the fact, that in the case of the Coulomb potential it is more convenient to study the integral representations of the considered quantum-mechanical functions than their integral equations. Furthermore, in this case it is more clearly seen that the Coulomb ambiguities are closely (immediately) connected to the problems of the Fourier transform.

*I am grateful to Grigor Giorgadze and Ilia Lomidze who have been collaborating with me on these topics.

Using the well-known definition of the Fourier transform [8]:

$$\psi_C^+(\vec{q},\vec{k}) = \int\limits_0^{+\infty} \exp(-i\vec{q}\vec{r} - \alpha r)\psi_C^+(\vec{r},\vec{k})d\vec{r}, \quad \alpha \to 0, \tag{1}$$

which is considered as the regularization (renormalization) procedure, we obtained new results, some of which will be described below.

2 Main results. Using the representation (1), we got the regular Fourier image of the two-particle Coulomb function in the weak sense.

The main result will be formulated as the following theorem.

Theorem 1. *The regular Coulomb function F_C^+ can be expressed as follows:*

$$F_C^+(\vec{q},\vec{k}) := H\,(\xi^2 - \lambda^2)\,\psi_C^+(\vec{q},\vec{k}) + \tilde{H}\,(\xi^2 - \lambda^2)\int\limits_\Omega \psi_C^+(\vec{q},\vec{k})f(\hat{q})d\hat{q}, \tag{2}$$
$$\xi = q - k, \quad 0 < \lambda << 1.$$

The Coulomb function ψ_C^+ is well-defined in the domain $q \neq k$, and in the region of the point $q \to k$ the specified function can be expressed as follows:

$$\psi_C^+(\vec{q},\vec{k}) = Z_C^+(q/k)\delta(\hat{q} - \hat{k}), \quad q \to k. \tag{3}$$

The regular radial part Z_C^+ in the expression (3) is defined in the weak sense.

Remark. The function $H\,(x)$ in the expression (2) is the Heaviside Unit function, which is determined in the following manner:

$$H(\xi^2 - \lambda^2) = \begin{cases} 1 & \xi^2 - \lambda^2 > 0 \\ 0 & \xi^2 - \lambda^2 \leq 0 \end{cases} = \begin{cases} 1 & |\xi\,| > \lambda, \\ 0 & |\xi\,| \leq \lambda, \end{cases} \quad \lambda > 0,$$
$$\xi = q - k, \quad 0 < \lambda << 1,$$

and the function $\tilde{H}(x)$ is defined as follows:

$$\tilde{H}(\xi^2 - \lambda^2) = 1 - H(\xi^2 - \lambda^2).$$

Let us note, that the parameter λ separates the small region $[\,k - \lambda, k + \lambda\,]$ of the point $q = k$ from the rest of the impulse space. □

Substituting the formula (3) into the second summand of the expression (2) and taking into account the following condition:

$$C\int\limits_\Omega \delta(\hat{p} - \hat{k})f(\hat{p})d\hat{p} = 1,$$

for the regular function (2) in the region of the point $q = k$, we obtain the following definition in the weak sense:

$$F_C^+(\vec{q}, \vec{k}) := Z_C^+(q/k), \quad q \to k. \tag{4}$$

As we already mentioned, the function Z_C^+ is determined in the weak representation. It is the well-defined infinitesimal function in the domain $|\tau| \leq \lambda$:

$$Z_C^+(q/k) := \tilde{\mathcal{O}}(\tau, \varphi),$$
$$q = k + \tau, \quad |\tau| \leq \lambda, \quad 0 < \lambda << 1,$$

which vanishes at the point $\tau = 0$.

The radial part of the function (2) can be formally expressed as follows:

$$F_{Cl}^+(q,k) := H(\xi^2 - \lambda^2)\psi_{Cl}^+(q,k) + \tilde{H}(\xi^2 - \lambda^2)Z_C^+(q/k). \tag{5}$$

The function (2) with the asymptotic (4) has the following important property.

Theorem 2. *The regular Coulomb function (2) satisfies the following two-particle homogeneous equation of the perturbation theory:*

$$F_C^+(\vec{q}, \vec{k}) = \frac{g}{k^2 - q^2 + i0} \int_0^\infty V_C(\vec{q} - \vec{p}) F_C^+(\vec{p}, \vec{k}) d\vec{p}, \tag{6}$$

where the Coulomb potential V_C has the logarithmic singularity.

3 Conclusions. By the above scenario we can conclude that the regular Coulomb quantum-mechanical function F_C^+ belongs to the $\mathbb{L}_2$ (Hilbert) space. Accordingly, represents the orthogonal complete set of the quantum-mechanical functions of the continuous spectrum. In addition, it is the exact solution of the two-particle homogeneous integral equation of the perturbation theory.

Thus, we have rigorously shown that the function which is defined by the representation (1) exists, and it is correct in the weak representation.

Note that the approach described above also provides for the regularization of the Coulomb T matrix [9] and potential, so that the systematic consistent mathematical theory of two charged particles of the continuous spectrum can be formulated.

The above calculations together with the results [9] will be very useful while studying the scattering problems of the few charged particles.

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Received 16.05.2024; revised 22.07.2024; accepted 30.08.2024.

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