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ON TWO MULTIDIMENSIONAL SYSTEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ⋆

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Abstract. Two multidimensional partial differential models are considered. The first one is based on Maxwell's well-known system of equations. The uniqueness of the solutions of the corresponding initial-boundary value problems, the convergence of a decomposition method and the finite-difference scheme are studied. The second multidimensional biological model is also considered. Algorithms of sum approximation and variable directions have been studied.

Keywords and phrases: Maxwell's system, unique solvability, finite-difference scheme, decomposition method, algorithm of sum approximation, variable directions method, convergence.

AMS subject classification (2010): 35B32, 35B40, 35K55, 35Q60, 65M06.

Two multidimensional systems of nonlinear partial differential equations (SNPDE) are considered. One model is based on Maxwel's well-known SNPDE [1]:

$$\frac{\partial U}{\partial t} = -rot\left(v_m rot H\right), \quad \frac{\partial \theta}{\partial t} = v_m \left(rot H\right)^2. \tag{1}$$

Maxwell's system (1) is complex and its investigation and numerical resolution still yield for special cases (see, for example, [2], [3] and references therein).

Note that, system (1) can be reduced to the following integro-differential form [4]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t \left|rot H\right|^2 d\tau\right) rot H\right], \tag{2}$$

where $a = a(S)$ is defined for $S \in [0, \infty)$.

Many works are devoted to the investigation of (1) systems and for integro-differential models (2), corresponding to (1) (see, for example, [2] - [11] and references therein).

Let Ω be a bounded domain in the n-dimensional Euclidean space R^n, with sufficiently smooth boundary $\partial\Omega$. In the domain $Q = \Omega \times (0, T)$ of the variables $(x, t) = (x_1, x_2, ..., x_n, t)$ let us consider the following first type initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a\left(\int_0^t \sum_{l=1}^{n}\left|\frac{\partial U}{\partial x_l}\right|^2\right)\frac{\partial U}{\partial x_i}\right] = f(x,t), \quad (x,t) \in Q, \tag{3}$$

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \quad U(x,0) = 0, \quad x \in \overline{\Omega}, \tag{4}$$

where T is a fixed positive constant, f is a given function of its arguments.

The problem (3), (4) is similar to the problems considered in [6] and [11]. It is proved, using the modified version of Galerkin's method and compactness arguments [12]:

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Theorem 1. *If $a(S)=1+S, f\in W_2^1(Q), f(x,0)=0$, then problem* (3), (4) *has a unique solution with the properties:*

$$U\in L_4\left(0,T;\dot W_4^1(\Omega)\right),\quad \frac{\partial U}{\partial t},\sqrt{T-t}\frac{\partial^2 U}{\partial t\partial x_i},\sqrt{\psi}\frac{\partial^2 U}{\partial x_i\partial x_j}\in L_2(Q),\quad i,j=1,...,n,$$

where $\psi\in C^\infty(\overline\Omega), \psi(x)>0$, for $x\in\Omega$; $\frac{\partial\psi}{\partial v}=0$, for $x\in\partial\Omega, v$ is the outer normal of $\partial\Omega$.

On $[0,T]$ let us introduce a net with mesh points denoted by $t_j=j\tau, j=0,1...,J$, with $\tau=T/J$. Let us construct additive averaged Rothe's type scheme for (3), (4):

$$\eta_i\frac{u_i^{j+1}-u_i^j}{\tau}=\frac{\partial}{\partial x_i}\left[\left(1+\tau\sum_{k=1}^{j+1}\sum_{l=1}^{n}\left|\frac{\partial u_i^k}{\partial x_l}\right|^2\right)\frac{\partial u_i^{j+1}}{\partial x_i}\right]+f_i^{j+1},\tag{5}$$

with homogeneous boundary and initial $u_i^0=u^0=0$ conditions, where $u_i^j(x), i=1,...,n$, $j=0,1,...,J-1$, are solutions of problem (5), and:

$$u^j=\sum_{i=1}^n\eta_i u_i^j(x),\quad \sum_{i=1}^n\eta_i=1,\quad \eta_i>0,\quad \sum_{i=1}^n f_i^j(x)=f^{j+1}(x)=f(x,t_{j+1}).$$

Theorem 2. *If problem* (3), (4) *has a sufficiently smooth solution, then the solutions of* (5) *converge to the solutions of problem* (3), (4) *and the following estimate is true*

$$\|u^j-U^j\|=O\left(\tau^{1/2}\right),\quad j=1,...,J.$$

The convergence of the finite-difference scheme for the one-dimensional case of systems (1) and equations (2) are also studied (see, for example, [2], [3], [8]).

Mathematical modeling of many applied problems leads to the following SNPDE:

$$\frac{\partial U}{\partial t}=\sum_{\alpha=1}^n\frac{\partial}{\partial x_\alpha}\left(V_\alpha\frac{\partial U}{\partial x_\alpha}\right),\quad \frac{\partial V_\alpha}{\partial t}=-V_\alpha+g_\alpha\left(V_\alpha\frac{\partial U}{\partial x_\alpha}\right),\quad \alpha=1,...,n.\tag{6}$$

If $n=2$ and g_α are given sufficiently smooth functions, $\gamma_0\le g_\alpha(\xi_\alpha)\le G_0$, and γ_0,G_0 are positive constants, then system (6) describes the biological model [13].

In the cylinder $Q=\Omega\times(0,T)$, where $\Omega=\{x=(x_1,...,x_n):0<x_\alpha<1,\alpha=1,...,n\}$, consider system (6), with the following initial and boundary conditions:

$$U(x,0)=U_0(x),\quad V_{\alpha 0}(x)=V_0(x),\quad x\in\overline\Omega,\quad \alpha=1,...,n,\tag{7}$$

$$U(x,t)=0,\quad (x,t)\in\partial\Omega\times[0,T].\tag{8}$$

Here $U_0,V_{\alpha0},g_\alpha$ are given sufficiently smooth functions, such that: $V_{\alpha0}(x)\ge\delta_0$, $|g'_\alpha(\xi_\alpha)|\le G_1,\xi\in R,\alpha=1,...,n$; where δ_0,G_1 are some positive constants.

Introducing the following grids, hereafter we will use the usual inner products and the norms [14] for the grid functions:

$$\overline\omega_{\alpha h}=\{x_{i_1...i_n}=(i_1h_1,...,i_{\alpha-1}h_{\alpha-1},(i_\alpha-1/2)h_\alpha,i_{\alpha+1}h_{\alpha+1},...,i_nh_n),$$

$$i_\alpha=1,...,M_\alpha,i_\beta=0,...,M_\beta,\beta\ne\alpha,M_\beta h_\beta=1,\beta=1,...,n\},$$

$$\omega_h=\Omega\cap\overline\omega_h,\gamma=\overline\omega_h\setminus\omega_h,\overline\omega_h=\omega_h\cup\gamma_h,$$

$$\omega_\tau=\{t_j=j\tau,j=0,...,J,J\tau=T\},\overline\omega_{h\tau}=\overline\omega_h\times\omega_\tau,\overline\omega_{\alpha h\tau}=\overline\omega_{\alpha h}\times\omega_\tau,\alpha=1,...,n.$$

Using the well-known approach and notations [14], [15], for all $\alpha = 1, ..., n$, let us correspond the variable directions type difference scheme to the problem (6) - (8):

$$u_{\alpha t} = \sum_{\beta=1}^{\alpha} \left(\widehat{v}_\beta \widehat{u}_{\beta\overline{x}_\beta}\right)_{x_\beta} + \sum_{\beta=\alpha+1}^{n} \left(\widehat{v}_\beta \widehat{u}_{\beta\overline{x}_\beta}\right)_{x_\beta}, \quad v_{\alpha t} = -\widehat{v}_\alpha + g_\alpha(v_\alpha u_{\alpha\overline{x}_\alpha}), \tag{9}$$

$$u_\alpha(x,0) = U_0(x), \quad x \in \overline{\omega}_h, \quad v_\alpha(x,0) = v_{\alpha 0}(x), \quad x \in \overline{\omega}_{\alpha h}, \tag{10}$$

$$u_\alpha(x,t) = 0, \quad (x,t) \in \gamma_h \times \omega_\tau. \tag{11}$$

In (9) the functions $u_\alpha, \alpha = 1, ..., n$, are defined on $\overline{\omega}_{h\tau}$, while the v_α on $\overline{\omega}_{\alpha h\tau}$.

For the sufficient smoothness of the exact solution $U, V_1, ..., V_n$ of the problem (6) - (8), each of the difference equations (9) approximates the corresponding differential equations (6) with order of $O(\tau + \sum_{\beta=1}^{n} h_\beta^2)$ and $O(\tau + h_\alpha^2)$ respectively.

Theorem 3. *If the differential problem* (6) - (8) *has a sufficiently smooth solution* $U, V_1, ..., V_n$*, then the solution of the scheme of the type of variable directions* (9) - (11) *is absolutely stable with respect to initial data and converges to the exact solution of problem* (6) - (8) *when* $\tau \to 0, h_\alpha \to 0, \alpha = 1, ..., n$*, and the following inequality holds*

$$\sum_{\alpha=1}^{n} \{\|(u_\alpha - U)_{\overline{x}_\alpha}]|_\alpha + \|v_\alpha - V_\alpha]|_\alpha\} \le O\left(\tau + \sum_{\alpha=1}^{n} h_\alpha^2\right).$$

On each segment $\triangle_k = [k, (k+1)\tau], k = 1, ..., J$, let us consider the following averaged model of sum approximation for initial-boundary value problem (6) - (8):

$$\eta_i \frac{\partial u_i^k}{\partial t} = \frac{\partial}{\partial x_i}\left(v_i^k \frac{\partial u_i^k}{\partial x_i}\right), \quad \frac{\partial v_i^k}{\partial t} = -v_i^k + g_i\left(v_i^k \frac{\partial u_i^k}{\partial x_i}\right), \tag{12}$$

$$u_i^0(x,0) = U_0(x), \quad v_i^0(x,0) = v_{i,0}(x), \tag{13}$$

$$u_i^k|_{x_i=0} = u_i^k|_{x_i=1} = 0, \quad u_i^k(x,t_k) = u^{k-1}(x,t_k), \quad v_i^k = v_i^{k-1}(x,t_k), \tag{14}$$

$$u^k(x,t) = \sum_{i=1}^{n} \eta_i u_i^k(x,t), \quad \eta_i > 0, \quad \sum_{i=1}^{n} \eta_i = 1. \tag{15}$$

Theorem 4. *If the differential problem* (6) - (8) *has a sufficiently smooth solution, then the solution of* (12) - (15) *converges to the exact solution when* $\tau \to 0$ *and*

$$\|u^k(t) - U(t)\| + \sum_{i=1}^{n} \|v_i^k(t) - V_i(t)\| = O(\tau^{1/2}).$$

We should note that some questions of construction and investigation of the variable direction schemes and the average model of sum approximation for multidimensional, as well as difference schemes for onedimensional cases for the (6) type systems are discussed in some other papers as well (see, for example, [16] - [19] and references therein).

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