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A SIZE STRUCTURED MODEL OF THE POPULATION BASED ON A STOCHASTIC GROWTH EQUATION

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Abstract. A stochastic growth equation is given whose solution density satisfies size-structured population growth equation and boundary conditions.

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Let $N(t) = \int_0^\infty u(x,t)dx$ be the decomposition of population by the size. Then in [1],[2] the optimal harvesting (catching) problem for the system

$$\partial_t u(x,t) + \partial_x(g(x)u(x,t)) + m(x)u(x,t) = -\mu(t)u(x,t), \tag{1}$$

$$u(x,0) = u_0(x), \tag{2}$$

$$g(0)u(0,t) = \int_0^\infty \beta(t,x)u(x,t)dx. \tag{3}$$

under revenue function

$$\int_0^T \int_0^\infty \mu(t)u(x,t)dxdt = \int_0^T \mu(t)N(t)dt \xrightarrow{\mu} \max$$

was studied. Here $m(x), \beta(t,x)$ denote natural mortality and fecundity of population of size x, $\mu(t)$ denotes fishing mortality and $g(x)$ is rate of growth of individuals.

Our aim is to show that the solution of (1)-(3) may be represented by the distribution function of a stochastic processes.

Let $\eta_k, k = 1,2,\ldots$ be i.i.d. nonnegative random variables with probability density $f(x), x \geq 0$, $\sigma_n = \sum_k^n \eta_k$ and let η_0 be an independent random variable with probability density $f_0(x)$. Denote $N(t) = \#\{n : \eta_0 + \sigma_n \leq t\}$. Suppose $A(t) = t - \sigma_{N_t}$ and $a(t,x) = P(A(t) \leq x)$.

A) Let $u(x) = \sum_{n=1}^\infty f^{n*}(x)$ and let f_0 be differentiable and $\tilde{u} = f_0 + u * f$, where $*$ denotes convolution.

Lemma 1. *Let condition A) be satisfied. Then the equation $E\Phi(A(t)) = \int_0^\infty \Phi(x)(1 - F(x))\tilde{u}(t-x)dx$ is satisfied for every continuous bounded function Φ.*

Proof. For each bounded, continuous function Φ we get

$$E\Phi(A_t) = \sum_{n=0}^{\infty} E\Phi(t-\tau_n)I_{(\tau_n \le t < \tau_{n+1})}$$

$$= \sum_{n=0}^{\infty} E\Phi(t-\tau_n)I_{(t_n \le t < \tau_{n+1})}(1-F(t-\tau_n))$$

$$= \sum_{n=0}^{\infty} \int_0^t \Phi(t-s)(1-F(t-s))dF^{*n}(s).$$

$$= \int_0^t \Phi(t-s)(1-F(t-s))dU(s) = \int_0^t \Phi(s)(1-F(s))u(t-s)ds.$$

□

Let $x(t)$ be the solution of the growth equation

$$\dot{x}(t) = g(x(t)), x(0) = 0.$$

Let $Y(t) = \begin{cases} x(t-\tau_{N_t}), & t < \zeta \\ \partial, & t \ge \zeta \end{cases}$, where $\zeta = \inf\{s; \int_0^s \mu(v)dv > \tau\}$ and τ independent, exponentially distributed r. v. with parameter 1. Then Y satisfies the equation

$$Y(t) = Y(0) + \int_0^t g(Y(s))ds - \int_0^t Y(s-)dN(s),\ t < \zeta.$$

For each bounded, continuous function φ one obtains

$$\int_0^\infty \varphi(r)\rho(t,r)dr = E\varphi(Y(t)), t < \zeta) = E\varphi(A(t)), t < \zeta)$$

$$= E\varphi(x(t-\tau_{N_t}))e^{-\int_0^t \mu(v)dv}.$$

Using lemma 1 for the function $\Phi(t) = \varphi(x(t))$ we get

$$E\varphi(x(t-\tau_{N_t}))e^{-\int_0^t \mu(v)dv} = e^{-\int_0^t \mu(v)dv} \int_0^t \Phi(s)(1-F(s))\tilde{u}(t-s)ds$$

$$= e^{-\int_0^t \mu(v)dv} \int_0^\infty \varphi(x(s))(1-F(s))\tilde{u}(t-s)/g(x(s))dx(s)$$

$$= e^{-\int_0^t \mu(v)dv} \int_0^\infty \varphi(r)(1-F(x^{-1}(r)))\tilde{u}(t-x^{-1}(r))/g(r)dr$$

and $\rho(t,r)e^{\int_0^t \mu(v)dv} = (1-F(x^{-1}(r)))\tilde{u}(t-x^{-1}(r))/g(r)$. Hence

$$\begin{aligned}
&\partial_t\rho(t,r)e^{\int_0^t \mu(v)dv} + \mu(t)\rho(t,r)e^{\int_0^t \mu(v)dv} \\
&+\partial_r(g(r)\rho(t,r))e^{\int_0^t \mu(v)dv} \\
&= \partial_t(\rho(t,r)e^{\int_0^t \mu(v)dv}) + \partial_r(g(r)\rho(t,r)e^{\int_0^t \mu(v)dv}) \\
&= (1-F(x^{-1}(r)))\tilde{u}'(t-x^{-1}(r))/g(r) \\
&-(1-F(x^{-1}(r)))\tilde{u}'(t-x^{-1}(r))x^{-1'}(r) \\
&-f(x^{-1}(r)))\tilde{u}(t-x^{-1}(r))x^{-1'}(r) \\
&= -f(x^{-1}(r)))\tilde{u}(t-x^{-1}(r))/g(r) \\
&= -\frac{f(x^{-1}(r)))}{1-F(x^{-1}(r))}e^{\int_0^t \mu(v)dv}\rho(t,r).
\end{aligned}$$

Integrating the equation $\partial_t\rho(t,r)+\mu(t)\rho(t,r)+\partial_r(g(r)\rho(t,r))+\frac{f(x^{-1}(r)))}{1-F(x^{-1}(r))}\rho(t,r)=0$ and using $\int_0^\infty \rho(t,r)dr = e^{-\int_0^t \mu(v)dv}$ one obtains

$$g(0)\rho(t,0) = \int_0^\infty \frac{f(x^{-1}(r))}{1-F(x^{-1}(r))}\rho(t,r)dr.$$

Finally we get

Proposition 1. *Let condition A) be satisfied. Then*

$$\begin{aligned}
\partial_t\rho(t,r)+\partial_r(g(r)\rho(t,r))+\mu(t)\rho(t,r)+\frac{f(x^{-1}(r)))}{1-F(x^{-1}(r))}\rho(t,r)=0, \\
g(0)\rho(t,0) = \int_0^\infty \beta(r)\rho(t,r)dr,
\end{aligned}$$

where $\beta(r) = \frac{1}{1-F(x^{-1}(r))}(f(x^{-1}(r))$.

Corollary. *The function $u(x,t) = N_0\rho(t,x)$ satisfies equations (3).*

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