Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 38, 2024

MATHEMATICAL MODEL OF DETONATION SHOCK WAVE PROPAGATION IN A NONHOMOGENEOUS STAR

Temur Chilachava Rusudan Zedashidze

Abstract. The mathematical modeling of astrophysics processes is one of the most actual problem of modern applied mathematics. To solve many problems of astrophysics, it is necessary to study the dynamics of gaseous bodies interacting with a gravitational field. The work considers a non-avtomodel problem about the central explosion of nonhomogeneous gas body (cubic function of star density drop from central core to the surface) bordering vacuum, which is in equilibrium in its own gravitational field. The solution of the problem in the vicinity behind the detonation shock wave (the fracturing surface of the first kind) is sought in the form of a singular asymptotic decomposition by a small parameter. Analytically, the main (zero) approximation for the law of motion and the thermodynamic characteristics of the medium was accurately found. The Cauchy problem for zero approximation of the law of motion of the detonation shock wave is solved exactly, with the help of Appell hypergeometric function of two variables. The asymptotics of the zero approximation of the law of the detonation shock wave is found at the moment and during the time coming on the surface of the object. The time of coming on the surface is also found.

Keywords and phrases: Nonhomogeneous star, gravitational field, explosion, detonation shock wave, singular decomposition, appel hypergeometric function.

AMS subject classification (2010): 97M50, 83F05, 76U05.

1 Introduction. To solve many problems of astrophysics, it is necessary to study the dynamics of the interaction of gas bodies with the gravitational field. The concept of studying celestial phenomena should be based on posing and solving a number of dynamic problems about the motion of gravitational gas, which are considered to be important mathematical models of stellar motion and evolution [1]. Then we solved several one-dimensional and two-dimensional nonautomodel problems about explosive processes in gas bodies and the spread of a detonation or shock wave to the surface of the body, followed by a separation into the vacuum [2,3]. In these mathematical models, we considered mainly homogeneous gas bodies.

2 Statement of the mixed problem for the system of nonlinear equations in partial derivatives. We will use the equations of the adiabatic spherical-symmetric motion of the gravitating gas in the Lagrangian form [2]

$$\frac{\partial^2 r}{\partial t^2} + 4\pi r^2 \frac{\partial p}{\partial m} + \frac{km}{r^2} = 0,\ p = (\gamma - 1) f(m)\rho^{\gamma},\ \rho = \left[4\pi r^2 \frac{\partial r}{\partial m}\right]^{-1} \tag{1}$$

here m is the mass of the $r(m,t)$ radius sphere, k is a gravity constant, γ is an adiabatic index, $f(m)$ function is associated with the distribution of entropy over the Lagrangian m coordinate, the $r = r(m,t)$ function defines the law of motion of the medium, $\frac{\partial r}{\partial t}$ is medium speed, $p(m,t)$ is medium pressure, $\rho(m,t)$ is medium density.

The integral energy equation for the gas layer enclosed between the surfaces $m = 0$ and $m = M(t)$ has the form:

$$T + U - kV = E + \int_0^t \left\{ \dot{M} \left[\frac{1}{2} \left(\frac{\partial r}{\partial t} \right)^2 + \frac{p}{(\gamma - 1)\rho} - \frac{kM}{R} + Q \right] - 4\pi r^2 \frac{\partial r}{\partial t} p \right\}_1 dt \quad (2)$$

$$T = \frac{1}{2} \int_0^M \dot{r}^2 \, dm, \; U = \frac{1}{\gamma_2 - 1} \int_0^M \frac{P}{\rho} \, dm, \; V = - \int_0^M \frac{m}{r} \, dm,$$

where T, U, V is kinetic, internal and potential (gravitational) gas energy, E is the explosion energy, $m = M(t)$ is law of detonation shock wave motion by mass, $R(t) = r(M(t), t)$ is detonation shock wave radius. Indices 1, 2 denote respectively the gas position before and after the surface of strong discontinuity.

If boundary conditions are solved with respect to parameters of the gas behind the wave we get the following:

$$\rho_2 = \frac{\gamma_2 + 1}{\gamma_2 - 1} \rho_1 \left[1 + \frac{1}{\gamma_2 - 1} \left(\frac{\gamma_2}{\gamma_1} \frac{a_1^2}{\left(\dot{R} - \left(\frac{\partial r}{\partial t} \right)_1 \right)^2} + 1 - g \right) \right]^{-1}, \; a_1^2 = \frac{\gamma_1 p_1}{\rho_1},$$

$$p_2 = \frac{1}{\gamma_2 + 1} \left[p_1 + \rho_1 \left(\dot{R} - \left(\frac{\partial r}{\partial t} \right)_1 \right)^2 (1 + g) \right], \quad (3)$$

$$\dot{R} - \left(\frac{\partial r}{\partial t} \right)_2 = \frac{1}{\gamma_2 + 1} \left(\dot{R} - \left(\frac{\partial r}{\partial t} \right)_1 \right) \left[\gamma_2 + \frac{\gamma_2}{\gamma_1} \frac{a_1^2}{\left(\dot{R} - \left(\frac{\partial r}{\partial t} \right)_1 \right)^2} - g \right],$$

$$g = \sqrt{ \left(1 - \frac{\gamma_2}{\gamma_1} \frac{a_1^2}{\left(\dot{R} - \left(\frac{\partial r}{\partial t} \right)_1 \right)^2} \right)^2 + \frac{2(\gamma_2 + 1)(\gamma_1 - \gamma_2) a_1^2}{\gamma_1 (\gamma_1 - 1) \left(\dot{R} - \left(\frac{\partial r}{\partial t} \right)_1 \right)^2} - \frac{2(\gamma_2^2 - 1) Q}{\left(\dot{R} - \left(\frac{\partial r}{\partial t} \right)_1 \right)^2} }.$$

Besides, the continuity of Euler's and Lagrange's variables ought to be taken into account $[r]_1^2 = 0, \; [m]_1^2 = 0$.

Initial conditions ($t = 0$, phone) determine the initial state of a gravitating gas sphere and are the exact solutions of system (1).

Thus, the initial-boundary problem is considered in the domain $\Omega = \{t \in (0, t_*), \; m \in (0, M(t))\}$, where $t = 0$ is the moment of explosion, t_* is the moment of time when the detonation wave comes out on the surface of the body. Boundary conditions on the external unknown boundary $m = M(t)$ are like (3) and in the center takes place $r(m,t) = 0, \ m = 0$.

3 Exact solution before and approximate solution after detonation shock wave. Let us discuss the problem of the central explosion at the $t = 0$ moment of a nonhomogeneous gas sphere (star) balanced in its own gravitation field.

Thus, the exact solution of the system of equations (1) that corresponds to the nonhomogeneous gas sphere balanced in its own gravitation field is taken as an initial condition.

The gravitation constant k, the sphere center density ρ_c and the sphere radius a are taken as main units of dimension

$$\rho = 1-r^3, m = 4\pi r^3\left(\frac{1}{3}-\frac{r^3}{6}\right), \frac{\partial r}{\partial t} = 0, p = 4\pi\left[\frac{1}{6}(1-r^2)-\frac{1}{10}(1-r^5)+\frac{1-r^8}{48}\right]. \quad (4)$$

Qualitative analysis of the system of equations (1) and boundary conditions (3) shows that the solution in the vicinity behind the detonation shock wave can be sought in the form of the next singular decomposition

$$r = R_0(\tau)+\varepsilon H(m,\tau)+..., \ R(\tau) = R_0(\tau)+\varepsilon R_1(\tau)+..., \quad (5)$$

$$p = p_0(m,\tau)+\varepsilon p_1(m,\tau)+..., \ \rho = \frac{\rho_0(m,\tau)}{\varepsilon}+\rho_1(m,\tau)+..., \ \tau = t/\sqrt{\varepsilon}, \ \varepsilon = \frac{\gamma_2-1}{\gamma_2+1} \ll 1.$$

Substituting (5) singular decomposition into the system of equations (1), integral equation (2) and boundary conditions (3), we get a zero approximation to the solution of the problem

$$p_0(m,\tau) = R_0'^2(\tau)(1-R_0^3(\tau))+\frac{R_0''(\tau)(M_0(\tau)-m)}{4\pi R_0^2(\tau)}, \ M_0(\tau) = \frac{2\pi}{3}R_0^3(\tau)(2-R_0^3(\tau)), \quad (6)$$

$$\rho_0(m,\tau) = p_0^{\frac{1}{\gamma_2}}(m,\tau)\left[R_0'^2(T_0(m))\right]^{-\frac{1}{\gamma_2}}\left[1+\frac{\left(\frac{1}{\gamma_2}-\gamma_*\right)a_1^2(m)+2Q_0}{R_0'^2(T_0(m))}\right]^{-1},$$

$$a_1^2(m) = \frac{4\pi\gamma_1}{1-r^3(m)}\left[\frac{1}{6}(1-r^2(m))-\frac{1}{10}(1-r^5(m))+\frac{1-r^8(m)}{48}\right],$$

$$\gamma_* \equiv \frac{1}{\gamma_1}, \ \gamma_1-1 = O(1), \ \gamma_* = \frac{\gamma_1-\gamma_2}{\gamma_1-1} = O(1), \ \gamma_1-1 = O(\varepsilon),$$

$$\gamma_* \equiv O, \ \gamma_1 = \gamma_2, \ Q = \frac{Q_0}{\varepsilon}, \ Q_0 = O(1),$$

where the function $r = r(m)$ is defined from the equation (4) and has the form

$$r(m) = \sqrt[3]{1-\sqrt{1-\frac{3m}{2\pi}}}.$$

The function $R_0(\tau)$ in (6) is the solution of the following Cauchy problem

$$\frac{\pi}{6}\left[1-R_0^3(\tau)\right]R_0'^2R_0^3(\tau)\left[2-R_0^3(\tau)\right]=E_0,\ R_0(0)=0 \tag{7}$$

and has the form

$$\int_0^{R_0}\sqrt{(1-R_0^3)(2-R_0^3)R_0^3}\,dR_0=\sqrt{\frac{6E_0}{\pi}}\tau_*,\ \int_0^x\sqrt{(1-x^3)(2-x^3)x^3}\,dx$$

$$=\left(x^4\left(\begin{array}{c}-45\sqrt{2-2x^3}\sqrt{2-x^3}x^3F_1\left(\frac{11}{6};\frac{1}{2},\frac{1}{2};\frac{17}{6};x^3,\frac{x^3}{3}\right)+\\ 132\sqrt{2-2x^3}\sqrt{2-x^3}F_1\left(\frac{5}{6};\frac{1}{2},\frac{1}{2};\frac{11}{6};x^3,\frac{x^3}{3}\right)+110(x^6-3x^3+2)\end{array}\right)\right)/$$

$$/\left(605\sqrt{x^3(x^6-3x^3+2)}\right)+constant, \tag{8}$$

$$F_1(\alpha;\beta;\beta';\gamma;x;y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}}x^my^n$$

is a hypergeometric function of two Appel variables,

$$\int_0^1\sqrt{(1-x^3)(2-x^3)x^3}\,dx\approx 0.357753,\ \tau_*\approx 0.357753\sqrt{\frac{\pi}{6E_0}}.$$

At that $\tau\to 0_+$ asymptotics of Cauchy's problem solution (8) are calculated, as well as asymptotics at, where $\tau\to\tau_{*-}$ (τ_* is time of detonation shock wave release to the sphere surface)

$$R_0(\tau)\cong(\frac{75E_0}{4\pi})^{\frac{1}{5}}\tau^{\frac{2}{5}},\ \tau\to 0_+,\ R_0(\tau)\cong 1-\left(\frac{9E_0}{2\pi}\right)^{\frac{1}{3}}(\tau_*-\tau)^{\frac{2}{3}},\ \tau\to\tau_{*-}.$$

REFERENCES

1. Sedov, L.I. Similarity and Dimensional Methods in Mechanics, 1993.
2. Golubyatnikov, A., Chilachava, T. Propagation of a detonation wave in a gravitating sphere with subsequent dispersion into a vacuum. *Fluid Dynamics*, **21**, 4 (1986), 673-677.
3. Chilachava, T. A central explosion in an inhomogeneous sphere in equilibrium in its own gravitational field. *Fluid Dynamics*, **23**, 3 (1988), 472-477.

Received 30.05.2024; revised 27.08.2024; accepted 25.09.2024.

Author(s) address(es):

Temur Chilachava, Rusudan Zedashidze
Sokhumi State University
Politkovskaya str. 61, 0186 Tbilisi, Georgia
E-mail: temo_chilachava@yahoo.com, rusudanzedashidze@yahoo.com