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THE BOUNDARY VALUE PROBLEM FOR ONE CLASS OF HIGHER-ORDER NONLINEAR HYPERBOLIC SYSTEMS

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Abstract. The boundary value problem for one class of higher-order nonlinear hyperbolic systems is considered. The theorems on existence, uniqueness and nonexistence of solutions of this problem are proved.

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On a plane of variables x and t consider the following fourth-order hyperbolic systems

$$\Box^2 u_i + f_i\left(u_1, \ldots, u_N\right) = F_i\left(x, t\right), \quad i = 1, \ldots, N, \tag{1}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$; $f = (f_1, \ldots, f_N)$ and $F = (F_1, \ldots, F_N)$ are given functions, while $u = (u_1, \ldots, u_N)$ is an unknown $N-$dimensional vector function, $N \geq 2$.

Denote by $D_T : 0 < x < t, \;\; t < T$ the angular domain bounded by characteristic segment $\gamma_{1,T} : x = t, \;\; 0 \leq t \leq T, \;\; \gamma_{2,T} : x = 0, \;\; 0 \leq t \leq T$ and $\gamma_{3,T} : t = T, \;\; 0 \leq x \leq T,$ temporal and spatial orientation segments, respectively.

For system (1) in the domain D_T consider the boundary value problem with the following statement: find in the domain D_T a solution $u = \left(u_1\left(x,t\right), \ldots, u_N\left(x,t\right)\right)$ to system (1) which on the boundary $\partial D_T = \gamma_{1,T} \cup \gamma_{2,T} \cup \gamma_{3,T}$ of the domain D_T satisfies the following homogeneous conditions

$$u|_{\gamma_{1,T}} = u(t,t) = 0, \quad 0 \leq t \leq T, \tag{2}$$

$$u|_{\gamma_{2,T}} = u(0,t) = 0, \quad u_x|_{\gamma_{2,T}} = u_x(0,t) = 0, \quad 0 \leq t \leq T, \tag{3}$$

$$u|_{\gamma_{3,T}} = u(x,T) = 0, \quad u_t|_{\gamma_{3,T}} = u_t(x,T) = 0. \tag{4}$$

It should be noted that in the scalar case, the Darboux type problem for the nonlinear equation (1) in the angular domain D_T, when the boundary conditions are given only on the $\gamma_{1,T}$ and $\gamma_{2,T}$ parts of the boundary of this domain, is discussed in [1]. Boundary value problems for higher$-$order nonlinear partial differential equations and systems with a different structure are studied in papers [2][4] (see also the literature cited in these papers).

Introduce the Hilbert space $\overset{o}{W}{}^{1}_{2,\Box}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{\overset{o}{W}{}^{1}_{2,\Box}(D_T)} = \int_{D_T} \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + (\Box u)^2 \right] dxdt \tag{5}$$

of the classical space

$$\overset{o}{C}{}^{k}\left(\overline{D}_T, \partial D_T\right) := \{u \in C^k\left(\overline{D}_T\right) : u|_{\gamma_{1,T}} = 0, \quad u|_{\gamma_{2,T}} = u_x|_{\gamma_{2,T}} = 0,$$

$$u|_{\gamma_{3,T}} = u_t|_{\gamma_{3,T}} = 0, \} \quad k \geq 1,$$

for k=2.

It follows from (5) that if $u \in \overset{o}{W}{}^{1}_{2,\Box}(D_T)$, then $u \in \overset{o}{W}{}^{1}_{2}(D_T)$ and $\Box u \in L_2(D_T)$. Here $W^1_2(D_T)$ is the well−known Sobolev space consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$, and $\overset{o}{W}{}^{1}_{2}(D_T) := \{u \in W^1_2(D_T) : u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory.

Below, on the nonlinear vector function $f = (f_1, \ldots, f_N)$ from (1) we impose the following requirements

$$f \in C\left(\mathbb{R}^N\right), \quad |f(u)| \leq M_1 + M_2 |u|^\alpha, \quad \alpha = const > 1, \quad u \in \mathbb{R}^N, \tag{6}$$

where $|\cdot|$ is the norm of the space $\mathbb{R}^N$, $M_i = const \geq 0$, $i = 1, 2$.

Definition 1. Let the vector function f satisfy the condition (6) and $F \in L_2(D_T)$. The vector function $u \in \overset{o}{W}{}^{1}_{2,\Box}(D_T)$ is said to be a weak generalized solution of the problem $(1) - (4)$, if for any vector function $\varphi = (\varphi_1, \ldots, \varphi_N) \in \overset{o}{W}{}^{1}_{2,\Box}(D_T)$ the integral equality

$$\int_{D_T} \Box u \Box \varphi dxdt + \int_{D_T} f(u) \varphi dxdt = \int_{D_T} F \varphi dxdt \quad \forall \varphi \in \overset{o}{W}{}^{1}_{2,\Box}(D_T) \tag{7}$$

is valid.

It is easy to verify, that the classical solution $u \in \overset{o}{C}{}^{4}\left(\overline{D}_T, \partial D_T\right)$ of the problem $(1)-(4)$ represents a weak generalized solution according to the Definition 1, i.e. it satisfies the integral identity (7), on the other hand, if the weak generalized solution of the problem $(1) - (4)$ belongs to the class $\overset{o}{C}{}^{4}\left(\overline{D}_T, \partial D_T\right)$, then it will be the classical solution of this problem.

Consider the following condition

$$\lim_{|u| \to \infty} \inf \frac{uf(u)}{|u|^2} \geq 0, \tag{8}$$

which concerns the behavior of the vector function f in a neighborhood of infinity, where

$$uf(u) = \sum_{i=1}^{N} u_i f_i(u), \quad |u|^2 = \sum_{i=1}^{N} u_i^2.$$

Lemma 1. *Let $F \in L_2(D_T)$ and let conditions (6) and (8) be fulfilled. Then for a weak generalized solution $u \in \overset{o}{W}{}^1_{2,\Box}(D_T)$ of the boundary value problem (1)(4) the following a priori estimate*

$$\|u\|_{\overset{o}{W}{}^1_{2,\Box}(D_T)} \leq c_1 \|F\|_{L_2(D_T)} + c_2$$

is valid, where the constants $c_1 > 0$ and $c_2 \geq 0$, independent of u and F.

The validity of the following theorem follows from Lemma 1 and the Leray−Schauder theorem.

Theorem 1. *Let the conditions (6) and (8) be fulfilled. Then for any vector function $F = (F_1, \dots, F_N) \in L_2(D_T)$ the boundary value problem (1)(4) has at least one weak generalized solution $u = (u_1, \dots, u_N)$ in the space $\overset{o}{W}{}^1_{2,\Box}(D_T)$.*

Consider the following condition imposed on the vector function f

$$(f(u) - f(v))(u - v) \leq 0 \quad \forall u, v \in \mathbb{R}^N. \tag{9}$$

Regarding the uniqueness of a weak generalized solution of the boundary value problem (1)(4), the following theorem is true.

Theorem 2. *Let the vector function f satisfy the conditions (6) and (8). Then for any vector function $F \in L_2(D_T)$ the boundary value problem $(1)-(4)$ cannot have more than one weak generalized solution $u \in \overset{o}{W}{}^1_{2,\Box}(D_T)$.*

The following theorem follows from Theorems 1 and 2.

Theorem 3. *Let the vector function f satisfy the conditions $(6), (8)$ and (9). Then for any vector function $F = (F_1, \dots, F_N) \in L_2(D_T)$ the problem $(1)-(4)$ has a unique weak generalized solution $u = (u_1, \dots, u_N)$ in the space $\overset{o}{W}{}^1_{2,\Box}(D_T)$.*

Now let us give one class of vector functions f, when the condition (6) is satisfied, but the condition (8) is violated, and in this case for a sufficiently wide class of vector functions $F = (F_1, \dots, F_N) \in L_2(D_T)$ the problem $(1)-(4)$ has no weak generalized solution. This class is given by the following formula

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^{N} a_{ij} |u_j|^{\beta_{ij}} + b_i, \quad i = 1, \dots, N, \tag{10}$$

where constants a_{ij}, βij and b_i satisfy the following inequalities

$$a_{ij} > 0, \quad \beta_{ij} = const > 1, \quad \sum_{i=1}^{N} b_i > 0, \quad i, j = 1, \dots, N. \tag{11}$$

The following theorem holds.

Theorem 4. *Let the vector function* $f = (f_1, \dots, f_N)$ *satisfy the conditions* (10) *and* (11), $F^0 = (F_1^0, \dots, F_N^0) \in L_2(D_T)$, $\sum\limits_{i=1}^{N} F_i^0 < 0$, *and* $F = \mu F^0$, $\mu = const > 0$. *Then there exists a number* $\mu_0 = \mu_0(a_{ij}, \beta_{ij}) > 0$ *such that the problem* $(1)-(4)$ *has no weak generalized solution* $u \in \overset{o}{W}{}^1_{2,\Box}(D_T)$, *when* $\mu > \mu_0$.

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