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ONE BOUNDARY PROBLEM FOR THE CIRCLE WITH QUADRUPLE POROSITY *

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Abstract. In this paper the linear theory of elasticity for quadruple porosity isotropic materials with macro-, meso-, micro-, and submicropores are considered. A two-dimensional system of equations of plane deformation is written in the complex form and its general solution is represented by means of three analytic functions of a complex variable and three solutions of Helmholtz equations. The concrete problem is solved for the circle.

Keywords and phrases: Quadruple porosity, plane deformation, the boundary problem.

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1 Introduction. The intended applications of the mathematical models for multiporosity (triple or quadruple) materials are to geological materials such as oil and gas reservoirs, rocks and soils, manufactured porous materials such as ceramics and pressed powders, geothermal reservoirs, and biomaterials such as bone.

M. A. Biot in his original paper presented the governing equations of poroelasticity for an isotropic single porosity material based on Darcy's law [1]. The linear theory for double porosity deformable materials by using Darcy's extended law is developed by Wilson and Aifantis [2]. The several new triple porosity mathematical models for solids with hierarchical macro-, meso-, and microporosity structure are presented by Straughan [3]. An extensive review of the results on the multi-porosity media may be found in the new books (Svanadze [4]). The basic boundary value problems for triple porosity materials models are studied in [5, 6].

2 The basic equations. Let (x_1, x_2, x_3) be a point of the Euclidean three dimensional space $\mathbb{R}^3$. The governing system of homogeneous equations of the linear theory of thermoelasticity for quadruple porosity isotropic materials with macro-, meso-, micro-, and submicropores may be written in the following equations [4]:

- Equations of motion

$$\partial_j t_{ji} = 0, \quad i, j = 1, 2, 3 \tag{1}$$

where t_{ij} is the component of the stress tensor, $\partial_j = \frac{\partial}{\partial x_j}$.

- Constitutive equations

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - (\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + \beta_4 p_4)\delta_{ij}, \quad k = 1, 2, 3, \tag{2}$$

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where λ and μ are the Lamé constants, $\beta_m,\ m=\overline{1,4}$ is the effective stress parameter, $p_1,\ p_2,\ p_3$ and p_4 are fluid pressures in the macro, meso, micro, and submicro scales, respectively, δ_{ij} is the Kronecker delta, e_{ij} is the component of strain tensor and defined by

$$e_{ij}=\frac{1}{2}\left(\partial_j u_i+\partial_i u_j\right), \tag{3}$$

where u_i are the components of the displacement vector in solid.

• From Equation of fluid mass conservation and Darcy's law

$$\Delta p - Ap = 0, \tag{4}$$

where $p=(p_1,p_2,p_3,p_4)^T$,

$$A=\begin{pmatrix} \dfrac{\gamma_{12}+\gamma_{13}+\gamma_{14}}{a_1} & -\dfrac{\gamma_{12}}{a_1} & -\dfrac{\gamma_{32}}{a_1} & -\dfrac{\gamma_{14}}{a_1} \\ -\dfrac{\gamma_{21}}{a_2} & \dfrac{\gamma_{21}+\gamma_{23}+\gamma_{24}}{a_2} & -\dfrac{\gamma_{23}}{a_2} & -\dfrac{\gamma_{24}}{a_2} \\ -\dfrac{\gamma_{31}}{a_3} & -\dfrac{\gamma_{31}}{a_3} & \dfrac{\gamma_{31}+\gamma_{32}+\gamma_{34}}{a_3} & -\dfrac{\gamma_{34}}{a_3} \\ -\dfrac{\gamma_{41}}{a_4} & -\dfrac{\gamma_{42}}{a_4} & -\dfrac{\gamma_{43}}{a_4} & \dfrac{\gamma_{41}+\gamma_{42}+\gamma_{43}}{a_4} \end{pmatrix}, \tag{5}$$

$a_l,\ l=\overline{1,4}$ is the effective stress parameter, $\gamma_{lm},\ m=\overline{1,4}$ is the fluid transfer rate between phase l and phase m.

3 The plane deformation.

In the case of plane deformation $u_3=0$ while the functions u_1, u_2, p_1, p_2, p_3 and p_4 do not depend on the coordinate x_3 [7].

On the plane Ox_1x_2, we introduce the complex variable $z=x_1+ix_2=re^{i\alpha},\ (i^2=-1)$ and the operators $\partial_z=0.5(\partial_1-i\partial_2)$, $\partial_{\bar z}=0.5(\partial_1+i\partial_2)$, $\bar z=x_1-ix_2$, and $\Delta=4\partial_z\partial_{\bar z}$.

From (1)-(5) we obtain the following system of equations of motion in the linear theory for porous materials with quadruple porosity expressed in terms of the components of the displacement vector field $u_1,\ u_2$, the change of pressures p_1, p_2, p_3, p_4 (in the complex form)

$$\begin{aligned} &\mu\Delta u_+ + 2(\lambda+\mu)\partial_{\bar z}\theta - 2\partial_{\bar z}(\beta_1 p_1+\beta_2 p_2+\beta_3 p_3+\beta_4 p_4)=0,\\ &\Delta p - Ap = 0. \end{aligned} \tag{6}$$

Theorem 1. *The general solution of the system (6) is represented as follows [7, 8]:*

$$\begin{aligned} &2\mu u_+ = \varkappa\varphi - z\overline{\varphi'(z)} - \overline{\psi(z)} + e_1\left(f(z)+z\overline{f'(z)}\right)\\ &+e_2\partial_{\bar z}\chi_1(z,\bar z)+e_3\partial_{\bar z}\chi_2(z,\bar z)+e_4\partial_{\bar z}\chi_3(z,\bar z),\\ &p_m = f'(z)+\overline{f'(z)}+l_{m1}\chi_1(z,\bar z)+l_{m2}\chi_2(z,\bar z)+l_{m3}\chi_3(z,\bar z),\quad m=\overline{1,4} \end{aligned} \tag{7}$$

where $\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}$, $\varphi(z)$, $\psi(z)$ *and* $f(z)$ *is the arbitrary analytic functions of a complex variable* z, $\chi_1(z,\bar z)$, $\chi_2(z,\bar z)$ *and* $\chi_3(z,\bar z)$ *are an arbitrary solution of the Helmholtz equation*

$$\Delta\chi_j - \kappa_j\chi_j = 0,\quad j=1,2,3,$$

κ_j is eigenvalues and $(l_{11},\ l_{21},\ l_{31},\ l_{41})$, $(l_{12},\ l_{22},\ l_{32},\ l_{42})$ and $(l_{13},\ l_{23},\ l_{33},\ l_{43})$ are eigenvectors of the matrix A and

$$e_1 = \frac{\nu(\beta_1+\beta_2+\beta_3+\beta_4)}{\lambda+2\mu}, \quad e_2 = \frac{4\mu(\beta_1 l_{11}+\beta_2 l_{21}+\beta_3 l_{31}+\beta_4 l_{41})}{(\lambda+2\mu)\kappa_1},$$

$$e_3 = \frac{4\mu(\beta_1 l_{12}+\beta_2 l_{22}+\beta_3 l_{32}+\beta_4 l_{42})}{(\lambda+2\mu)\kappa_2}, \quad e_2 = \frac{4\mu(\beta_1 l_{13}+\beta_2 l_{23}+\beta_3 l_{33}+\beta_4 l_{43})}{(\lambda+2\mu)\kappa_3}.$$

4 The Dirichlet problem for the circle. Let us consider the elastic circle with quadruple porosity bounded by the circumference of radius R (Fig. 1). The origin of coordinates is at the center of the circle [7].

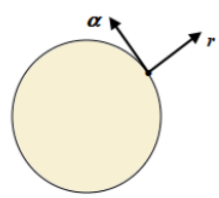

Figure 1: The elastic circle.

On the circumference, we consider the following boundary value problem

$$u_r + iu_\alpha = A, \quad p_m = B^m, \quad r = R, \tag{8}$$

where A and B^m are sufficiently smooth functions.

The analytic functions $\varphi(z)$, $\psi(z)$, $f(z)$ and the metaharmonic functions $\chi_1(z,\bar{z})$, $\chi_2(z,\bar{z})$, $\chi_3(z,\bar{z})$ are represented as the series [7]

$$\begin{aligned} &\varphi(z) = \sum_{n=1}^{\infty} a_n z^n, \ \ \psi(z) = \sum_{n=0}^{\infty} b_n z^n, \ \ \chi_1(z,\bar{z}) = \sum_{-\infty}^{+\infty} \alpha_n I_n(\gamma r) e^{in\vartheta}, \\ &f(z) = \sum_{n=1}^{\infty} c_n z^n, \ \ \chi_2(z,\bar{z}) = \sum_{-\infty}^{+\infty} \beta_n I_n(\gamma r) e^{in\vartheta}, \ \ \chi_3(z,\bar{z}) = \sum_{-\infty}^{+\infty} \gamma_n I_n(\gamma r) e^{in\vartheta}, \end{aligned} \tag{9}$$

where $I_n(\cdot)$ are the modified Bessel functions of the first kind of n-th order.

Expand the function $A/2\mu \cdot e^{i\alpha}$ and B^m given on $r = R$, in a complex Fourier series

$$\frac{A}{2\mu} e^{i\alpha} = \sum_{-\infty}^{\infty} A_n e^{in\alpha}, \quad B^m = \sum_{-\infty}^{\infty} B_n^m e^{in\alpha}. \tag{10}$$

Substituting (7), (9), (10) into the boundary conditions (8) and comparing the coefficients of $e^{in\alpha}$ we have

$$\varkappa R a_1 - R\overline{a}_1 + e_1 R c_1 + e_1 R\overline{c}_1 + \frac{e_2\sqrt{\kappa_1}}{2} I_1(\sqrt{\kappa_1}R)\alpha_0 + \frac{e_3\sqrt{\kappa_2}}{2} I_1(\sqrt{\kappa_2}R)\beta_0$$

$$
\begin{aligned}
&+\frac{e_4\sqrt{\kappa_3}}{2}I_1(\sqrt{\kappa_3}R)\gamma_0 = A_1,\\
&-(n+2)R^{n+2}a_{n+2} - R^n b_n + e_1(n+2)R^{n+2}c_{n+2} + \frac{e_2\sqrt{\kappa_1}}{2}I_n(\sqrt{\kappa_1}R)\alpha_{n+1}\\
&+\frac{e_3\sqrt{\kappa_2}}{2}I_n(\sqrt{\kappa_2}R)\beta_{n+1} + \frac{e_4\sqrt{\kappa_3}}{2}I_n(\sqrt{\kappa_3}R)\gamma_{n+1} = \overline{A}_{-n}, \quad n \geq 0\\
&\varkappa R^n a_n + e_1 R^n c_n + \frac{e_2\sqrt{\kappa_1}}{2}I_n(\sqrt{\kappa_1}R)\alpha_{n-1} + \frac{e_3\sqrt{\kappa_2}}{2}I_n(\sqrt{\kappa_2}R)\beta_{n-1}\\
&+\frac{e_4\sqrt{\kappa_3}}{2}I_n(\sqrt{\kappa_3}R)\gamma_{n-1} = \overline{A}_n, \quad n \geq 2\\
&I_n(\gamma R)\alpha_n - k_3(n+1)R^{n+1}a_{n+1} + k_4(n+1)R^{n+1}c_{n+1} = B_n, \quad n \geq 0\\
&c_1 + \bar{c}_1 + l_{m1}I_0(\sqrt{\kappa_1}R)\alpha_0 + l_{m1}I_0(\sqrt{\kappa_2}R)\beta_0 + l_{m1}I_0(\sqrt{\kappa_3}R)\gamma_0\\
&= B_0^m, \quad m = \overline{1,4},\\
&(n+1)R^n c_{n+1} + l_{m1}I_n(\sqrt{\kappa_1}R)\alpha_n + l_{m1}I_n(\sqrt{\kappa_2}R)\beta_n + l_{m1}I_n(\sqrt{\kappa_3}R)\gamma_n\\
&= B_n^m, \quad n \geq 1, \quad m = \overline{1,4}.
\end{aligned}
\tag{11}
$$

From system (11) we can find all coefficients $a_n,\ b_n,\ c_n,\ \alpha_n,\ \beta_n,\ \gamma_n$.

It is easy to prove the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

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