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A NEW HAMILTONIAN SEMI-ANALYTICAL APPRAOCH TO VIBRATION ANALYSIS OF ELASTIC MULTI-LAYER COMPOSITE PLATES ⋆

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Abstract. This contribution addresses the development of a new *semi-analytical Hamiltonian* method for the *vibration* analysis of elastic *multi-layer (ML) composite* plates. By performing a *Legendre Transform*, the classical (displacements-based) Lagrangian functional is recast into a Hamiltonian one, so that the latter has (additionally to the displacements) the *transverse stresses* as arguments. The proposed method overcomes the inherent limitations of the traditional State Space Method when dealing with non-simply-supported boundary conditions such as clamped or free ones. The resulting transcendental eigenvalue problem is solved by adopting the Wittricks and Williams algorithm. To assess the robustness of the proposed method, the latter's numerical accuracy is investigated by analysing the free-vibration of elastic ML composite plates. Results are verified by comparison with published numerical or analytical ones.

Keywords and phrases: Vibration, plate, Hamiltonian, semi-analytical, Wittrick-Williams algorithm.

AMS subject classification (2010): 34B09, 35A15, 37N15, 70J10, 70J50, 74H45, 4K20, 74S05.

1 Introduction.

Plate structures are commonly found in civil, mechanical or aerospace engineering problems. In particular, plates made of laminated composite materials dominate today's technology. For the latter's analysis, displacement-based formulations are considered unsuitable due to anisotropy effects and interface continuity constraints (ICC). The here proposed approach tackles these issues by means of a Hamiltonian partially-mixed variational formulation (VF) where both the displacements and transverse stresses are used as independent variables. The gist of the strategy is to turn the plates partial differential equations into a numerically-discretized weak form in the in-plane coordinates while maintaining their analytically-solved strong form in the thickness one so that the ICC are satisfied exactly. However, in free-vibration analysis, the equations become transcendental once the ICC are considered. The Wittrick-Williams algorithm is carried out to overcome this difficulty, out of which the number of eigenvalues in a given interval can be counted so that none of them is missed during the analysis.

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2 Hamiltonian Dynamic Partially-Mixed Variational Formulation. In the context of the forced harmonic vibration analysis, the time *variable* is transformed into a frequency *parameter* ω so that the associated dynamic virtual work principle expresses as

$$\int_\Omega \left[\delta\mathscr{E}(\underline{\mathbf{u}}) - \rho\omega^2\delta\underline{\mathbf{u}}^\intercal\underline{\mathbf{u}}\right] d\Omega - \int_{\Gamma_F} \delta\underline{\mathbf{u}}^\intercal\underline{\mathbf{F}} d\Gamma = 0 \tag{1}$$

where Ω denotes the 3D bounded body, Γ_F the part of its boundary where the excitation vector $\underline{\mathbf{F}}$ applies and $\mathscr{E}(\underline{\mathbf{u}}) = \frac{1}{2}\underline{\varepsilon}^\intercal\underline{\underline{\mathbf{C}}}\,\underline{\varepsilon}$ refers to the elastic strain energy density with $\underline{\varepsilon}$ (resp. $\underline{\underline{\mathbf{C}}}$) the engineering linearized strain vector (resp. elastic stiffness matrix). The latter is split into thickness (z), $\underline{\varepsilon}_z = \{\gamma_{xz}, \gamma_{yz}, \varepsilon_{zz}\}^\intercal$ and in-plane (p), $\underline{\varepsilon}_p = \{\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}\}^\intercal$, contributions so that the strains-displacements relations write as

$$\underline{\varepsilon}_z = \dot{\underline{\mathbf{u}}} + \underline{\underline{\mathbf{D}}}_1\underline{\mathbf{u}}; \quad \underline{\varepsilon}_p = \underline{\underline{\mathbf{D}}}_2\underline{\mathbf{u}}; \quad \dot{\underline{\mathbf{u}}} = \frac{\partial\underline{\mathbf{u}}}{\partial z}; \quad \underline{\underline{\mathbf{D}}}_1 = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix}; \quad \underline{\underline{\mathbf{D}}}_2 = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix} \tag{2}$$

This leads to a similar matrix decomposition of the 3D elastic constitutive equations

$$\left\{ \begin{array}{c} \underline{\sigma}_p \\ \underline{\sigma}_z \end{array} \right\} = \begin{bmatrix} \underline{\underline{\mathbf{C}}}^*_{pp} & \underline{\underline{\mathbf{C}}}^*_{pz} \\ \underline{\underline{\mathbf{C}}}^{*\intercal}_{pz} & \underline{\underline{\mathbf{C}}}^*_{zz} \end{bmatrix} \left\{ \begin{array}{c} \underline{\varepsilon}_p \\ \underline{\varepsilon}_z \end{array} \right\} \tag{3}$$

with $\underline{\sigma}_p = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}^\intercal$, $\underline{\sigma}_z = \{\sigma_{xz}, \sigma_{yz}, \sigma_{zz}\}^\intercal$ the in-plane and transverse stress vectors.

Using Eq.(2) and Eq.(3) into the strain energy density, its explicit expression becomes

$$\begin{aligned} \mathscr{E}(\underline{\mathbf{u}}, \dot{\underline{\mathbf{u}}}) = &\frac{1}{2}\underline{\mathbf{u}}^\intercal\underline{\underline{\mathbf{D}}}_2^\intercal\underline{\underline{\mathbf{C}}}^*_{pp}\underline{\underline{\mathbf{D}}}_2\underline{\mathbf{u}} + \underline{\mathbf{u}}^\intercal\underline{\underline{\mathbf{D}}}_2^\intercal\underline{\underline{\mathbf{C}}}^*_{pz}\underline{\underline{\mathbf{D}}}_1\underline{\mathbf{u}} + \frac{1}{2}\underline{\mathbf{u}}^\intercal\underline{\underline{\mathbf{D}}}_1^\intercal\underline{\underline{\mathbf{C}}}^*_{zz}\underline{\underline{\mathbf{D}}}_1\underline{\mathbf{u}} + \\ &\underline{\mathbf{u}}^\intercal\underline{\underline{\mathbf{D}}}_2^\intercal\underline{\underline{\mathbf{C}}}^*_{pz}\dot{\underline{\mathbf{u}}} + \underline{\mathbf{u}}^\intercal\underline{\underline{\mathbf{D}}}_1^\intercal\underline{\underline{\mathbf{C}}}^*_{zz}\dot{\underline{\mathbf{u}}} + \frac{1}{2}\dot{\underline{\mathbf{u}}}^\intercal\underline{\underline{\mathbf{C}}}^*_{zz}\dot{\underline{\mathbf{u}}} \end{aligned} \tag{4}$$

The partially-mixed VF requires the definition of the dual variable associated to $\dot{\underline{\mathbf{u}}}$ which is

$$\underline{\mathbf{p}} = \frac{\partial\mathscr{E}}{\partial\dot{\underline{\mathbf{u}}}} = \underline{\underline{\mathbf{C}}}^{*\intercal}_{pz}\underline{\underline{\mathbf{D}}}_2\underline{\mathbf{u}} + \underline{\underline{\mathbf{C}}}^*_{zz}\underline{\underline{\mathbf{D}}}_1\underline{\mathbf{u}} + \underline{\underline{\mathbf{C}}}^*_{zz}\dot{\underline{\mathbf{u}}} \equiv \underline{\sigma}_z \tag{5}$$

The elimination of $\dot{\underline{\mathbf{u}}}$ in terms of $\underline{\mathbf{u}}$ and $\underline{\mathbf{p}}$ from Eq.(4), thanks to Eq.(5), in combination with the Legendre-Fenchel transformation gives the dynamic partially-mixed VF

$$\delta\int_\Omega \left[\underline{\mathbf{p}}^\intercal\dot{\underline{\mathbf{u}}} - \mathscr{H}(\underline{\mathbf{u}}, \underline{\mathbf{p}})\right] d\Omega + \int_{\Gamma_F} \delta\underline{\mathbf{u}}^\intercal\underline{\mathbf{F}} d\Gamma = 0 \tag{6}$$

where

$$\mathscr{H}(\underline{\mathbf{u}}, \underline{\mathbf{p}}) = \frac{1}{2}\underline{\mathbf{p}}^\intercal\underline{\underline{\mathbf{C}}}_{nn}\underline{\mathbf{p}} - \frac{1}{2}\underline{\mathbf{u}}^\intercal\underline{\underline{\mathbf{D}}}_2^\intercal\underline{\underline{\mathbf{C}}}_{pp}\underline{\underline{\mathbf{D}}}_2\underline{\mathbf{u}} - \underline{\mathbf{p}}^\intercal\underline{\underline{\mathbf{C}}}^\intercal_{pn}\underline{\underline{\mathbf{D}}}_2\underline{\mathbf{u}} - \underline{\mathbf{p}}^\intercal\underline{\underline{\mathbf{D}}}_1\underline{\mathbf{u}} + \frac{1}{2}\rho\omega^2\underline{\mathbf{u}}^\intercal\underline{\mathbf{u}} \tag{7}$$

and

$$\underline{\underline{\mathbf{C}}}_{pp} = \underline{\underline{\mathbf{C}}}^*_{pp} - \underline{\underline{\mathbf{C}}}^*_{pn}\underline{\underline{\mathbf{C}}}^{*^{-1}}_{nn}\underline{\underline{\mathbf{C}}}^{*\intercal}_{pn} \quad ; \quad \underline{\underline{\mathbf{C}}}_{nn} = \underline{\underline{\mathbf{C}}}^{*^{-1}}_{nn} \quad ; \quad \underline{\underline{\mathbf{C}}}_{pn} = \underline{\underline{\mathbf{C}}}^*_{pn}\underline{\underline{\mathbf{C}}}^{*^{-1}}_{nn} \tag{8}$$

The numerical discretization restricted to the in-plane space starts with $\underline{\mathbf{u}}(x,y,z) = \underline{\underline{\mathbf{N}}}(x,y)\underline{\tilde{\mathbf{u}}}(z)$ and $\underline{\mathbf{p}}(x,y,z) = \underline{\underline{\mathbf{N}}}(x,y)\underline{\tilde{\mathbf{p}}}(z)$, where $\underline{\underline{\mathbf{N}}}(x,y)$ contains the high-order spectral element shape functions. For free-vibration analysis, the discretization of Eq.(6) thus transforms into this first-order differential system $\underline{\dot{\boldsymbol{\Psi}}}(z) = \underline{\underline{\mathbf{H}}}\,\underline{\boldsymbol{\Psi}}(z)$, $\underline{\boldsymbol{\Psi}}(z) = \left\{\underline{\tilde{\mathbf{u}}}, \underline{\tilde{\mathbf{p}}}\right\}^{\mathsf{T}}$ whose solution is a matrix exponential. A state vectors equation on the layer's bottom (b) and top (t) is then obtained as $\underline{\boldsymbol{\Psi}}^t = \underline{\underline{\mathbf{S}}}\,\underline{\boldsymbol{\Psi}}^b$, leading to this Dynamic Stiffness System (DSS)

$$\left\{\begin{array}{c}\underline{\mathbf{p}}_b\\ \underline{\mathbf{p}}_t\end{array}\right\} = \begin{bmatrix}\underline{\underline{\mathbf{K}}}^{bb} & \underline{\underline{\mathbf{K}}}^{bt}\\ \underline{\underline{\mathbf{K}}}^{tb} & \underline{\underline{\mathbf{K}}}^{tt}\end{bmatrix}\left\{\begin{array}{c}\underline{\mathbf{u}}_b\\ \underline{\mathbf{u}}_t\end{array}\right\} \quad \text{where} \quad \underline{\underline{\mathbf{K}}} = \begin{bmatrix}-\underline{\underline{\mathbf{S}}}_{up}^{-1}\underline{\underline{\mathbf{S}}}_{uu} & \underline{\underline{\mathbf{S}}}_{up}^{-1}\\ \underline{\underline{\mathbf{S}}}^{pu} - \underline{\underline{\mathbf{S}}}^{pp}\underline{\underline{\mathbf{S}}}_{up}^{-1}\underline{\underline{\mathbf{S}}}^{uu} & \underline{\underline{\mathbf{S}}}_{pp}\underline{\underline{\mathbf{S}}}_{up}^{-1}\end{bmatrix} \tag{9}$$

Finally, the ML plate DSS is built by assembling each layer's one while enforcing the ICC. The Wittrick-Williams algorithm can then be used for solving the resulting eigenvalue problem.

3 Numerical results. A 2×2 mesh with various numbers of Gauss Legendre Lobatto grid points (up to 6×6 per element) is retained for the considered plate spectral element modeling (see Figure 1).

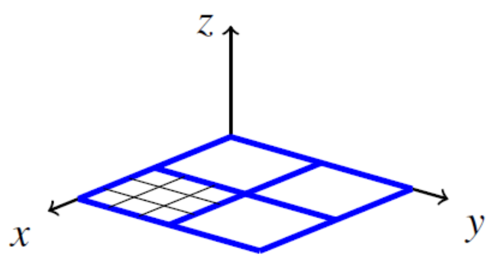

Figure 1: A2×2 in-plane mesh.

Table 1 shows the mesh refining convergence of the non-dimensional first eigenvalue $\tilde{\omega} = \omega\frac{a^2}{\pi^2}\sqrt{\frac{\rho h}{D_0}}$, $D_0 = E_2 h^3/12(1-\nu_{12}\nu_{21})$, for a all edges clamped (CCCC) $[0°/90°/0°]$ laminated square plate having $E_1 = 40E_2$, $G_{12} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = 0.25$, $E_2 = 1\,\text{GPa}$ and $S = \frac{a}{h} = \frac{10}{1}$. Table 2 shows the first 5 non-dimensional eigenvalues for the same but cantilever (CFFF) plate.

4 Conclusions. A new semi-analytical partially-mixed Hamiltonian approach for the vibration analysis of ML composite plates was proposed. The displacement and transverse stresses are the independent variables and ICC are satisfied exactly. Free-vibration analysis of CCCC and CFFF square ML plates demonstrates the excellent performance of the proposed methodology. As a perspective, it is worthy to extend the latter to piezoelectric smart plates.

Table 1 – First non-dimensional eigenvalue $\tilde{\omega}$ for a [0°/90°/0°] laminated CCCC square plate

Solutions	Mesh	Eigenvalue	Error with respect to [2] (%)
DQM* [1]	15×15	7.432	–
FSDT*-based Ritz [2]	13×13	7.411	–
Present semi-analytical	2×2	8.037	*8.45*
	4×4	7.463	*0.70*
	5×5	7.450	*0.53*
	6×6	7.450	*0.53*

* DQM : Differential Quadrature Method; FSDT: First-order Shear Deformation Theory

Table 2 – First 5 non-dimensional eigenvalues $\tilde{\omega}$ for a [0°/90°/0°] laminated square CFFF plate

Solutions (in-plane mesh)	Mode	1	2	3	4	5
DQM [1] (15×15)		1.919	2.105	4.175	7.757	7.996
FSDT-based Ritz [2] (13×13)		1.918	2.103	4.188	7.757	7.961
Present semi-analytical (6×6)		1.898	2.085	4.182	7.763	7.970
Error with respect to [2] (%)		*-1.04*	*-0.86*	*-0. 14*	*0.08*	*0.11*

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