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ON THE REALIZATION OF THE GAUSS-HERMITE APPROXIMATE METHOD FOR THE CAUCHY PROBLEM

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Abstract. This paper represents the part of a cycle of works, dedicated to the solution of the Cauchy problem for ordinary differential equations with high order of accuracy.

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In [1] we considered the class of schemes, corresponding to the Gauss-Hermite type method for the approximate solution of the Cauchy problem for ordinary differential equations (ODEs) with high order of accuracy. This method is a realization of the extrapolation method [2] which proposes some clear extensions for the investigation of the initial-boundary value problems. Below we give some remarks and corresponding observation of some important publications, dedicated to the construction of the initial data for the Cauchy problem. It is well known that there are two classes of methods by which the initial data are defined for the numerical solution of the Cauchy problem by high order of the remainder term: Runge-Kutta-Buthcher type and Adams-Bashforth methods. When the number of nodes tends to infinity, the necessary condition of stability and convergence is broken for Runge-Kutta-Buthcher type methods since the coefficients (using for calculate right part of ODEs) $b_{j,s}$, $j = 1, ..., m$, $s = 1, ..., m-1$ (m is order of schemes) for definition the corresponding process have different signs [3, Tabbele 15.5, p.329]. For Adams-Bashfort type methods, the condition of stability and convergence depends on the sign of weights of Newton-Kotes quadrature formulas [4]. More precisely, we proved that an error estimate (the difference between exact and approximate solutions) depends also on the sum of absolute value of the weights of quadrature formulas. It is known that Kuzmin's effect about unbounded of corresponding weights is true [5]. The author proves that the sums of positive or negative coefficients tend to infinity. Thus, for both classes they are unstable if using high order of approximations when the number of nodes tends to infinity. If we use the methods, containing two stages as in [2], for the second stage when the Gauss type convergence process is applicable, the corresponding subschemes are stable. At the same time the process of stability defining on the subintervals for the first stages for local error estimates are not investigated, when the number of nodes tends to infinity.

In [1] we created the method of the approximate solution of the Cauchy problem for ODEs, containing two stages: first, for local processes using Hermite interpolation formulas (depending on ordinates and slopes); and the second containing Gauss type quadrature formulas. The roots of the Legendre polynomials have been taken as the nodes. Below we modify this process.

At first we introduced matrices of nodes in Fejer's sense [6]. Let us consider the Hermite interpolation formula with ordinates and slopes:

$$H_{2n-1}(x) = \sum_{k=1}^{n} y_k A_k^{(n)}(x) + \sum y'_k B_k^{(n)}(x), \tag{1}$$

where

$$A_k^{(n)} = \left[1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k)\right] l_k^2(x), \quad B_k^{(n)} = (x - x_k) l_k^2(x),$$

$$l_k(x) = l_k^{(n)}(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \quad \omega(x) = \omega_n(x) = \prod_{k=1}^{n}(x - x_k).$$

Let us consider the linear functions:

$$v_k(x) = v_k^{(n)}(x) = 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k). \tag{2}$$

Following to Leopold Fejer we introduce the definition: If for all $x \in (a, b)$ and for all $k,\ n$ $(n = 1, 2, 3, ...,\ k = 1, 2, ..., n)$ the functions

$$v_k^{(n)}(x) \geq 0,$$

then matrix $\{x_k^{(n)}\ (k = 1, 2, ..., n,\ n = 1, 2, 3, ...)\}$ is called as normal matrix. If for same conditions

$$v_k^{(n)}(x) \geq \rho > 0,$$

then the corresponding matrix is called a strong normal or ρ-normal matrix. Below we define $x_k^{(n)}$ nodes as the roots of the polynomial:

$$\omega(x) = \omega_n(x) = \int_{-1}^{x} P_{n-i}(t)dt, \tag{3}$$

where $P_{n-1}(x)$ are Legendre polynomials. In [6, p. 556] it is proved that $\{x_k^{(n)}\ (k = 1, 2, ..., n,\ n = 1, 2, 3, ...)\}$ is a 1-normal matrix. We also consider the case when the nodes are the roots of Chebyshev polynomials. The corresponding matrix is a 1/2 normal [see 6, p. 556].

Now we consider the of approximate solution of the Cauchy problem:

$$y'(x) = f(x, y(x)), \quad 0 < x \leq l, \tag{4}$$

$$y(0) = y_0. \tag{5}$$

We applied here Gauss-Hermite type schemes when nodes are defined by (3) or roots of Chebyshev polynomials.

We define the process of finding the approximate solution of (4), (5) by using [1] and either Lobatto or Curtis-Clanshow quadrature formulas. For the realization of this scheme it is necessary to prepare the initial table with high order of accuracy. As we noted above, Bakhvalov's method [4] is unstable. Now we modify this approach.

Let for application of (1) the first interval be $(0, l_1)$ with nodes as elements of ρ normal matrices. For application of (1) we have to know approximately $y = y_i$ $(i = 0, 1, ..., n)$ and $y_i' = f(x_i, y_i)$ Below we find the initial data with high order of accuracy. For this we assume that problem (4), (5) is solved by any rough numerical scheme and for the difference between exact and approximate solutions we have: $\eta(x) = h^p\varphi(x),\ \varphi(x) \in C(0.l)$.

Now from [4] equations (4), (5) we have:

$$y(x_j) = y_1 + \sum_{i=1}^{n} c_{ji} y'(x_i) + R_j, \quad (j = 2, 3, ..., n). \tag{6}$$

By neglecting the remainder, we get:

$$\zeta(x_j) = y(x_j) - \overline{y(x_j)} = \sum_{i=1}^{n} c_{ji}(y'(x_i) - \overline{y'(x_i)}) + R_j. \tag{7}$$

Now we solve (7) by the iteration method. We define $|\zeta(x)| = z(x)$ and consider the following relations (k is a number of iterations):

$$z^{[k]}(x_j) \le \sum_{i=1}^{n} |c_{ji}| M z^{[k-1]}(x_i),\ M = \sup\left|\frac{\partial f}{\partial y}\right|. \tag{8}$$

For estimating $z^{[k]}(x_j)$ below we use some inequalities [see e.g. 6]:

$$\sum_{i=1}^{n} |c_{ji}| \le \lambda_n x_j,\ \lambda_n \le \frac{1}{\rho}\sqrt{n},$$

where λ_n are Lebesgue numbers and let $\eta(x)$ correspond to $k = 0$. Then we have:

$$z^{[1]}(x_j) \le h^p M_1,\ M_1 = \sup|\varphi(x)|,\ z^{[k+1]}(x_j) \le \left(\frac{x_n}{\rho}\sqrt{n}M\right)^k h^p M_1,$$

$$\zeta(x_j) \le z^{[k]}(x_j) \le ch^{2n}.$$

Now we extend (1) for the interval $(0, 2l_1)$ and calculate $y_i,\ y_i'$ $(i = n, n+1, ..., 2n-1)$. We underline that the nodes are choose so that they would be the elements of ρ-normal matrix for interval $(l_1, 2l_1)$ too. Now for finding the values $y_n,\ y_{2n}$ we applied Lobatto type quadrature processes, as x_i $(i = 1, 2, ..., n)$ roots of $\omega(x)$ are also the abscissas of corresponding quadrature formulas. Let us recall that [see e.g. 7, point 25.4.32]:

$$\int_{-1}^{1} f(x)dx = \frac{2}{n(n-1)}[f(-1) + f(1)] + \sum_{i=2}^{n-1} w_i f(x_i) + R_n,$$

$$w_i = \frac{2}{n(n-1)[P_{n-1}(x_i)]^2}, \ (x_i \neq \pm 1):$$

$$P'_{n-1}(x_i) = 0, \ i = 2, ..., n-1; \ R_n = \frac{-n(n-1)^2 2^{n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^2} f^{(2n-2)}(\xi), \ (-1 < \xi < 1).$$

This is Lobatto's formula with weights, abscissas and remainder.

Now let us formulate the final results by using the above data. At first if we rewrite the Hermite formula (1) for interval $(sl_1, \ (s+1)l_1), \ s = 0, 1, 2, ..., m, \ ml_1 = l$ the following priory estimation is valid:

$$|H_{2n-1}(x)| \leq \max |y_i| + \max |y'_i| \frac{l_1}{\rho}.$$

On the another hand, if we apply Lobatto's or other Gauss type quadrature formulas (same of [1]) we see that the calculate of approximate solution of the Cauchy problem is a stable process and is converging with order $O(h^{2n})$ corresponding to the remainder members of Hermite and Gauss type relations.

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