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ON THE SPACES OF SPHERICAL POLYNOMIALS AND GENERALIZED THETA-SERIES WITH QUADRATIC FORMS OF FIVE VARIABLES

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Abstract. The spherical polynomials of order $\nu = 2$ with respect to quadratic form of five variables are constructed and the basis of the space of these spherical polynomials is established. The space of generalized theta-series with respect to the quadratic form of five variables is considered. The basis of this space is constructed.

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1 Introduction.

Let

$$Q(X) = Q(x_1, \cdots, x_r) = \sum_{1\le i\le j\le r} b_{ij}x_ix_j$$

be an integer positive definite quadratic form of r variables and let $A = (a_{ij})$ be the symmetric $r \times r$ matrix of quadratic form $Q(X)$, where $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, for $i < j$. Let A_{ij} denote the cofactor to the element a_{ij} in A and a^*_{ij} is the element of the inverse matrix A^{-1}.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1\le i,j\le r} a^*_{ij}\Big(\frac{\partial^2 P}{\partial x_i \partial x_j}\Big) = 0 \tag{1}$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [1]), and

$$\vartheta(\tau, P, Q) = \sum_{n\in\mathbb{Z}^r} P(n)z^{Q(n)}, \qquad z = e^{2\pi i\tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im}\tau > 0$$

is the corresponding generalized r-fold theta-series.

Let $P(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$. Hecke [2] calculated the dimension of the space $P(\nu, Q)$, $\dim P(\nu, Q) = \binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ and form the basis of the space of spherical polynomials of second order with respect to $Q(X)$.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.,

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \;:\; P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [1] calculated the dimension of the vector space $T(\nu, Q)$ for reduced binary quadratic forms Q. Gaigalas [3] gets the upper bounds for the dimension of the space $T(4, Q)$ and $T(6, Q)$ for some diagonal quadratic forms. In [4-6] we established the upper bounds for the dimension of the space $T(\nu, Q)$ for some quadratic forms of r variables, when $r = 3, 4, 5$, in a number of cases we calculated the dimension and form the bases of these spaces.

In this paper we form the basis of the space of spherical polynomials of second order $P(2, Q)$ with respect to some diagonal quadratic form $Q(X)$ of five variables and obtained the basis of the space of generalized theta-series $T(2, Q)$ with spherical polynomial P of second order and diagonal quadratic form Q of five variables.

2 The basis of the space $P(2, Q)$ and $T(2, Q)$.

Let

$$Q(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}x_4^2 + b_{55}x_5^2,$$

where $0 < b_{11} < b_{22} < b_{33} < b_{44} < b_{55}$ is a quadratic form of five variables. For these forms

$$D = \det A = 2^5 b_{11}b_{22}b_{33}b_{44}b_{55}, \qquad a_{ii}^* = \frac{1}{2b_{ii}}, \qquad a_{ij}^* = 0.$$

Let

$$P(X) = P(x_1, x_2, x_3, x_4, x_5) = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j} a_{kijl}x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_4^{j-l}x_5^{l}$$

be a spherical function of order ν with respect to the positive quadratic form $Q(x_1, x_2, x_3, x_4, x_5)$ of five variables and let

$$L = [a_{0000}, a_{1000}, a_{1100}, a_{1110}, a_{1111}, a_{2000}, \dots, a_{\nu\nu\nu\nu}]^T$$

be the column vector, where a_{kijl} $(0 \le l \le j \le i \le k \le \nu)$ are the coefficients of the polynomial $P(X)$.

We have [6], that $\dim P(\nu, Q) = \binom{\nu+4}{4} - \binom{\nu+2}{4}$. The polynomials (the coefficients of polynomial P_i are given in the brackets)

$$P_1(a_{0000}^{(1)}, a_{1000}^{(1)}, \dots, a_{\nu-2,\nu-2,\nu-2,\nu-2}^{(1)}, 1, 0, 0, \dots, 0),$$

$$P_2(a_{0000}^{(2)}, a_{1000}^{(2)}, \dots, a_{\nu-2,\nu-2,\nu-2,\nu-2}^{(2)}, 0, 1, 0, \dots, 0),$$

. . . .

$$P_t(a_{0000}^{(t)}, a_{1000}^{(t)}, \dots, a_{\nu-2,\nu-2,\nu-2,\nu-2}^{(t)}, 0, 0, 0, \dots, 1),$$

where the first $\binom{\nu+2}{4}$ coefficients from a_{0000} to $a_{\nu-2,\nu-2,\nu-2,\nu-2}$ are calculated through other $\binom{\nu+4}{4} - \binom{\nu+2}{4} = t$ coefficients, form the basis of the space $P(\nu, Q)$.

The basis spherical polynomials of second order ($\nu = 2$) with respect to $Q(x_1, x_2, x_3, x_4, x_5)$ have the form:

$$P_1 = a^{(1)}_{0000} x_1^2 + x_1x_2, \qquad P_2 = a^{(2)}_{0000} x_1^2 + x_1x_3, \qquad P_3 = a^{(3)}_{0000} x_1^2 + x_1x_4,$$

$$P_4 = a^{(4)}_{0000} x_1^2 + x_1x_5, \qquad P_5 = a^{(5)}_{0000} x_1^2 + x_2^2, \qquad P_6 = a^{(6)}_{0000} x_1^2 + x_2x_3,$$

$$P_7 = a^{(7)}_{0000} x_1^2 + x_2x_4, \qquad P_8 = a^{(8)}_{0000} x_1^2 + x_2x_5, \qquad P_9 = a^{(9)}_{0000} x_1^2 + x_3^2,$$

$$P_{10} = a^{(10)}_{0000} x_1^2 + x_3x_4, \qquad P_{11} = a^{(11)}_{0000} x_1^2 + x_3x_5, \qquad P_{12} = a^{(12)}_{0000} x_1^2 + x_4^2,$$

$$P_{13} = a^{(13)}_{0000} x_1^2 + x_4x_5, \qquad P_{14} = a^{(14)}_{0000} x_1^2 + x_5^2.$$

Using (1) for P_i we have

$$P_5 = -\frac{b_{11}}{b_{22}} x_1^2 + x_2^2, \quad P_9 = -\frac{b_{11}}{b_{33}} x_1^2 + x_3^2, \quad P_{12} = -\frac{b_{11}}{b_{44}} x_1^2 + x_4^2, \quad P_{14} = -\frac{b_{11}}{b_{55}} x_1^2 + x_5^2$$

and for other P_i the coefficient $a^{(i)}_{0000} = 0$. They form the basis of the space of spherical polynomials of second order with respect to $Q(x)$ and $\dim P(2, Q) = \binom{6}{4} - \binom{4}{4} = 14$.

Now we construct the corresponding generalized theta-series.

$$\vartheta(\tau, P_5, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} P_5(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} (-\frac{b_{11}}{b_{22}} x_1^2 + x_2^2) \Big) z^n$$

$$-2\frac{b_{11}}{b_{22}} z^{b_{11}} + \cdots + 2z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \ldots,$$

$$\vartheta(\tau, P_9, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} P_9(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} (-\frac{b_{11}}{b_{33}} x_1^2 + x_3^2) \Big) z^n$$

$$-2\frac{b_{11}}{b_{33}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 2z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 0z^{b_{55}} + \ldots,$$

$$\vartheta(\tau, P_{12}, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} P_{12}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} (-\frac{b_{11}}{b_{44}} x_1^2 + x_4^2) \Big) z^n$$

$$-2\frac{b_{11}}{b_{44}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 2z^{b_{44}} + \cdots + 0z^{b_{55}} + \ldots,$$

$$\vartheta(\tau, P_{14}, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} P_{14}(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} (-\frac{b_{11}}{b_{55}} x_1^2 + x_5^2) \Big) z^n$$

$$-2\frac{b_{11}}{b_{55}} z^{b_{11}} + \cdots + 0z^{b_{22}} + \cdots + 0z^{b_{33}} + \cdots + 0z^{b_{44}} + \cdots + 2z^{b_{55}} + \ldots,$$

These generalized theta-series are linearly independent since the determinant constructed from the coefficients of these theta-series is not equal to zero. In [6] we have $\dim T(\nu, Q) \le \binom{\frac{\nu}{2}+3}{3}$ and for $\nu = 2$ we have $\dim T(2, Q) \le 4$. Hence this theta-series form the basis of the space $T(2, Q)$. We have the following

Theorem 1. $\dim T(2,Q)=4$ *and the generalized theta-series:*

$$\vartheta(\tau,P_5,Q);\vartheta(\tau,P_9,Q);\vartheta(\tau,P_{12},Q);\vartheta(\tau,P_{14},Q)$$

form the basis of the space $T(2,Q)$.

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