Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 31, 2017

BOUNDARY VALUE PROBLEMS OF THE THEORY OF ELASTICITY OF POROUS COSSERAT MEDIA FOR SOLIDS WITH TRIPLE-POROSITY ⋆

Bakur Gulua Roman Janjgava

Abstract. The purpose of this paper is to consider the two-dimensional version of the linear theory of elasticity for solids with triple-porosity in the case of an elastic Cosserat medium. Using the analytic functions of a complex variable and solutions of the Helmholtz equation basic boundary value problems are solved explicitly for the circle.

Keywords and phrases: Triple-porosity, the stress tensor, a circle.

AMS subject classification (2010): 74F10, 74G05.

1 The plane deformation. Basic equations. In this paper we consider the two-dimensional version of the linear theory of elasticity for solids with triple-porosity in the case of an elastic Cosserat medium [1-4].

Let D be a a circle with the radius R. Let us assume that the domain D is filled with an isotropic material with triple-porosity [5, 6]. The basic homogeneous system of equations in the full coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows

$$\partial_\alpha \sigma_{\alpha\beta} = 0, \quad \partial_\alpha \mu_{\alpha 3} + (\sigma_{12} - \sigma_{21}) = 0, \quad (\alpha, \beta = 1, 2) \tag{1}$$

$$\begin{aligned} &\sigma_{\alpha\alpha} = -\beta_i p_i + \lambda\theta + 2\mu\partial_\alpha u_\alpha, \quad \sigma_{12} = (\mu+\alpha)\partial_1 u_2 + (\mu-\alpha)\partial_2 u_1 - 2\alpha\omega, \\ &\sigma_{21} = (\mu+\alpha)\partial_2 u_1 + (\mu-\alpha)\partial_1 u_2 + 2\alpha\omega, \quad \mu_{\alpha 3} = (\nu+\beta)\partial_\alpha \omega, \quad \theta := \partial_1 u_1 + \partial_2 u_2, \end{aligned} \tag{2}$$

where $\sigma_{\alpha\beta}$ are stress tensor components, $\mu_{\alpha 3}$ are moment stress tensor components, u_α are components of the displacement vector, p_i $(i = 1, 2, 3)$ are the pressures in the fluid phase, λ and μ are the Lamé parameters, $\alpha,\ \beta,\ \mu$ are the constants characterizing the microstructure of the considered elastic medium, β_i $(i = 1, 2, 3)$ are the effective stress parameters. In the stationary case, the values $p = (p_1, p_2, p_3)^T$ satisfy the following equation

$$\Delta p - Ap = 0, \quad A = \begin{pmatrix} b_1/a_1 & -a_{12}/a_1 & -a_{13}/a_1 \\ -a_{21}/a_2 & b_2/a_2 & -a_{23}/a_2 \\ -a_{31}/a_3 & -a_{32}/a_3 & b_3/a_3 \end{pmatrix} \tag{3}$$

where $a_i = \frac{k_i}{\mu'}$ (for the fluid phase, each phase i carries its respectively permeability k_i, μ' is fluid viscosity), a_{ij} is the fluid transfer rate between phase i and phase j, Δ is the 2D Laplace operator, $b_1 = a_{12} + a_{13}$, $b_2 = a_{21} + a_{23}$, $b_3 = a_{31} + a_{32}$.

⋆This work was supported by a financial support of Shota Rustaveli National Science Foundation (Grant SRNSF/FR/358/5-109/14).

On the plane x_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar z} = 0.5(\partial_1 + i\partial_2)$, $\bar z = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar z}$.

If relations (2) are substituted into system (1), then system (1) is written in the complex form

$$\begin{aligned} &2(\mu+\alpha)\partial_{\bar z}\partial_z u_+ + (\lambda+\mu-\alpha)\partial_{\bar z}\theta - 2\alpha i\partial_{\bar z}\omega - \partial_{\bar z}(\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3) = 0,\\ &2(\nu+\beta)\partial_{\bar z}\partial_z\omega + \alpha i(\theta - 2\partial_{\bar z}u_+) - 2\alpha\omega = 0, \quad (u_+ = u_1 + iu_2). \end{aligned} \tag{4}$$

2 The general solution of system (3)-(4). In this section, we construct the analogues of the Kolosov-Muskhelishvili formulas [7] for system (4).

Equations (3) imply that

$$p_i = f'(z) + \overline{f'(z)} + l_{i1}\chi_1(z,\bar z) + l_{i2}\chi_2(z,\bar z),$$

where $f(z)$ is an arbitrary analytic functions of a complex variable z in the domain V and $\chi_\alpha(z,\bar z)$ is an arbitrary solution of the Helmholtz equation $\Delta\chi_\alpha(z,\bar z) - \kappa_\alpha\chi_\alpha(z,\bar z) = 0$, κ_α are eigenvalues and (l_{11}, l_{21}, l_{31}), (l_{12}, l_{22}, l_{32}) are eigenvectors of the matrix A.

Theorem 1. *The general solution of the system of equations (4) is represented as follows:*

$$2\mu u_+ = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \delta^*(f'(z) + \overline{f'(z)}) + \frac{4\mu}{\lambda+2\mu}\partial_{\bar z}[\delta_1\chi_1(z,\bar z) + \delta_2\chi_2(z,\bar z)],$$

$$2\mu\omega = \frac{2\mu}{\nu+\beta}\chi(z,\bar z) - \frac{\kappa+1}{2}i(\varphi'(z) + \overline{\varphi'(z)}),$$

where $\kappa = \frac{\lambda+3\mu}{\lambda+\mu}$, $\delta^* = \frac{\mu(\beta_1+\beta_2+\beta_3)}{\lambda+2\mu}$, $\delta_\alpha := \frac{l_{1\alpha}}{\kappa_\alpha}\beta_1 + \frac{l_{2\alpha}}{\kappa_\alpha}\beta_2 + \frac{l_{3\alpha}}{\kappa_\alpha}\beta_3$, $\varphi(z)$ *and* $\psi(z)$ *are arbitrary analytic functions of a complex variable* z *in the domain* V, $\chi(z,\bar z)$ *is an arbitrary solution of the Helmholtz equation* $4\partial_z\partial_{\bar z}\chi(z,\bar z) - \xi^2\chi(z,\bar z) = 0$, $\xi^2 := \frac{2\mu\alpha}{(\nu+\beta)(\mu+\alpha)} > 0$.

3 A problem for a circle. In this section, we solve a concrete boundary value problem for a circle of radius R (Figure 1). On the boundary of the considered domain the values of pressures p_1 and p_2 and the displacement vector are given.

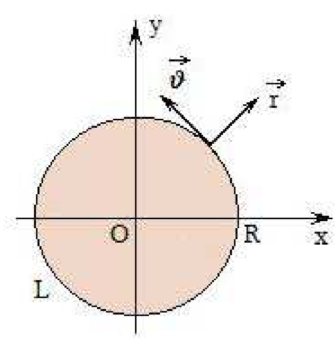

Figure 1: The circle

We consider the following problem

$$p_j|_{r=R} = P_j = \sum_{-\infty}^{+\infty} A_{nj}e^{in\vartheta}, \ A_{nj} = \overline{A}_{-nj}, \ j = \overline{1,3}, \tag{5}$$

$$\begin{aligned} &2\mu u_+|_{r=R} = 2\mu(G_1 + iG_2) = \sum_{n=-\infty}^{+\infty} B_n e^{in\vartheta},\\ &2\mu\omega|_{r=R} = G_3 = \sum_{n=-\infty}^{+\infty} C_n e^{in\vartheta}, \ C_n = \overline{C}_{-n}. \end{aligned} \tag{6}$$

The analytic function $f(z)$ and the metaharmonic functions $\chi_\alpha(z,\bar z)$ is represented as a series

$$f(z)=\sum_{n=1}^{+\infty}a_ne^{in\vartheta},\quad \chi_\alpha(z,\bar z)=\sum_{-\infty}^{+\infty}\alpha_{n\alpha}I_n(k_\alpha r)e^{in\vartheta},$$

where $I_n(\cdot)$ is a modified Bessel function of n-th order, and are substituted in the boundary conditions (5) we have

$$\sum_{n=1}^{+\infty}nR^{n-1}\left(a_ne^{i(n-1)\vartheta}+\bar a_ne^{-i(n-1)\vartheta}\right)$$
$$+\sum_{-\infty}^{+\infty}[l_{j1}\alpha_{n1}I_n(k_1R)+l_{j2}\alpha_{n2}I_n(k_2R)]e^{in\vartheta}=\sum_{-\infty}^{+\infty}A_{nj}e^{in\vartheta}.$$

Compare the coefficients at identical degrees. We obtain the following systems of equations

$$\begin{aligned}a_1+\bar a_1+l_{j1}I_0(k_1R)\alpha_{01}+l_{j2}I_0(k_2R)\alpha_{02}=A_{0j},\quad j=1,2,3,\\ nR^{n-1}a_n+l_{j1}I_{n-1}(k_1R)\alpha_{n-1\,1}+l_{j2}I_{n-1}(k_2R)\alpha_{n-1\,2}=A_{n-1j},\quad n>1.\end{aligned}\tag{7}$$

From (7) we can find $a_1+\bar a_1,\ a_n,\ \alpha_{n1},\ \alpha_{n2}$

Now the analytic functions $\varphi(z)$, $\psi(z)$ and the metaharmonic functions $\chi(z,\bar z)$ are represented as the series

$$\varphi(z)=\sum_{n=1}^{\infty}b_nz^n,\quad \psi(z)=\sum_{n=0}^{\infty}c_nz^n,\quad \chi(z,\bar z)=\sum_{-\infty}^{+\infty}\beta_nI_n(\zeta r)e^{in\vartheta}$$

and are substituted in the boundary conditions (6) we have

$$\sum_{n=1}^{\infty}(\kappa b_n+\delta^*)R^ne^{in\vartheta}-(\bar b_1-\delta^*\bar a_1)Re^{i\vartheta}-\sum_{n=0}^{\infty}(n+2)(\bar b_{n+2}-\delta^*\bar a_{n+2})R^{n+2}e^{-in\vartheta}$$
$$-\sum_{n=0}^{\infty}R^n\bar c_ne^{-in\vartheta}+\zeta i\sum_{-\infty}^{\infty}\beta_{n-1}I_n(\zeta R)e^{in\vartheta}=\sum_{-\infty}^{+\infty}A'_ne^{in\vartheta},$$
$$\frac{2\mu}{\nu+\beta}\sum_{-\infty}^{\infty}\beta_nI_n(\zeta R)e^{in\vartheta}-\frac{\kappa+1}{2}i\sum_{n=0}^{\infty}R^n[b_{n+1}e^{in\vartheta}-\bar b_{n+1}e^{-in\vartheta}]=\sum_{-\infty}^{+\infty}C_ne^{in\vartheta},$$

where $A'_n=B_n-\frac{4\mu}{\lambda+2\mu}\left[\frac{k_1\delta_1}{2}I_n(k_1R)+\frac{k_2\delta_2}{2}I_n(k_2R)\right]$.

Compare the coefficients at identical degrees. We obtain

$$R(\kappa b_1-\bar b_1)+\zeta iI_1(\zeta R)\beta_0=A''_1,\quad R(\kappa\bar b_1-b_1)-\zeta iI_1(\zeta R)\beta_0=\bar A''_1,$$

$$-\frac{\kappa+1}{2}iR(b_1-\bar b_1)+\frac{2\mu}{\nu+\beta}I_0(\zeta R)\beta_0=C_0,\quad \left(A''_1=A'_1-\delta^*R(a_1+\bar a_1)\right),$$

$$\kappa R^nb_n+i\zeta I_n(\zeta R)\beta_{n-1}=A'_n-\delta^*R^na_n,$$

$$\frac{2\mu}{\nu+\beta} I_{n-1}(\zeta R)\beta_{n-1} - \frac{\kappa+1}{2} inR^{n-1}b_n = C_{n-1}, \tag{8}$$
$$(n+2)R^{n+2}b_{n+2} + \zeta i\beta_{n-1}I_n(\zeta R) + R^n c_n = (n+2)\delta^* R^{n+2}a_{n+2} - \bar{A}'_{-n}, \ \ n \geq 0.$$

From (8) we can find coefficients $b_n,\ c_n,\ \beta_n$.

It is easy to prove the absolute and uniform convergence of the series obtained in the circle (including the contours) when the functions set on the boundaries have sufficient smoothness.

Similarly the problem can be solved when on the boundary of the considered domain the values of stresses are given.

REFERENCES

1. Beskos, D.E., Aifantis, E.C. On the theory of consolidation with double porosity-II. *Int. J. Engng. Sci.*, **24** (1986), 1697-1716.

2. Khaled, M.Y., Beskos, D.E., Aifantis, E.C. On the theory of consolidation with double porosity-III, A finite element formulation. *Int. J. for Numerical and Analytical Methods in Geomechanics*, **8**, 2 (1984), 101-123.

3. Liu, C.Q. Exact solution of unsteady axisymmetrical two-dimensional flow through triple-porosity media, *Applied Mathematics and Mechanics* **4**, 5 (1983), 717-724.

4. Cosserat, E., Cosserat, F. Theorie des Corps Deformables. *Hermann, Paris* 1909.

5. Janjgava, R. Elastic equilibrium of porous Cosserat media with double porosity. *Adv. Math. Phys.*, (2016), Article ID 4792148, 9 pages http://dx.doi.org/10.1155/2016/4792148.

6. Tsagareli, I., Bitsadze, L. Explicit solution of one boundary value problem in the full coupled theory of elasticity for solids with double porosity. *Acta Mech.* **226**, 5 (2015), 1409-1418.

7. Muskhelishvili, N.I. Some Basic Problems of the Mathematical Theory of Elasticity. Fundamental equations, plane theory of elasticity, torsion and bending. *Noordhoff International Publishing, Leiden*, 1977.

Received 14.05.2017; revised 15.09.2017; accepted 21.10.2017.

Author(s) address(es):

Bakur Gulua
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
Sokhumi State University
Anna Politkovskaia str. 9, 0186 Tbilisii, Georgia
E-mail: bak.gulua@gmail.com

Roman Janjgava
I. Vekua Institute of Applied Mathematics of
I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: roman.janjgava@gmail.com