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BASIC GROUPS OPERATIONS ON EXPONENTIAL MR-GROUPS

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Abstract. In the paper it is proved that the tensor completion is commutative with the operations of direct product and direct limit of exponential MR-groups and, but in general, is not commutative with the Cartesian product and the inverse limit of exponential MR-groups.

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The notion of an exponential R-group (R is an arbitrary associative ring with identity) was introduced by R. Lyndon in [1]. In [2] A. G. Myasnikov refined the notion of a exponential R-group by introducing an additional axiom. In particular, the new notion of exponential R-group is a direct generalization of the notion of a R-module to the case of noncommutative groups. With in honour to A. G. Myasnikov, R-group this axiom in M. Amaglobeli's paper [3] has homed MR-groups. Systematic study of MR-groups has started in [4, 5, 6, 7, 8, 9]. Note that the results of the study were very useful for solving the well-known problems of Tarski. In paper [2] it is shown that in the investigation of exponential MR-group the decisive role is played by the notion of tensor completion. In this paper we investigate the problem of the commutability of a functor of tensor completion with basic groups operations.

1 Basic notions in the theory of exponential MR-groups. Recall the basic definitions (see [1, 2]). Let R be an arbitrary associative ring with identity and let G be a group. Fix an action of the ring R on G, i.e. a map $G \times R \to G$. The result of the action of $\alpha \in R$ on $g \in G$ is written as g^α. Consider the following axioms:

(i) $g^1 = g$, $g^0 = e$, $e^\alpha = e$ $(1 \in R,\ e \in G)$;

(ii) $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;

(iii) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;

(iv) $[g,h] = e \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha$ (MR-axiom).

Definition 1 ([1]). The group G is called an **exponential R-group** (or **R-group**) after Lyndon if an action of the ring R on G satisfying axioms (i)–(iii) is given.

Definition 2 ([2]). The group G is called an **exponential R-group** (or **MR-group**) if an action of the ring R on G satisfies axioms (i)–(iv). Then R is called a ring of scalars of the group G.

Let $\mathfrak{L}_R$ and $\mathfrak{M}_R$ be the classes of all exponential R-group after Lyndon and all MR-groups. $\mathfrak{L}_R \supseteq \mathfrak{M}_R$. Any Abelian MR-group is an R-module, and vice versa. There exist Abelian Lyndon R-groups that are not R-modules (see [10], where the structure of free abelian R-group is extensively investigated), i.e. $\mathfrak{L}_R \supset \mathfrak{M}_R$.

Most of natural examples of exponential R-groups lie in $\mathfrak{M}_R$. For example, unipotent groups over a field K of zero characteristic are MR-groups, pro-p-groups are exponential $\mathsf{M}\mathbb{Z}_p$-groups over the ring $\mathbb{Z}_p$ of p-adic integers, etc (see [2] for examples).

Definition 3 ([2])**.** A homomorphism of R-groups $\varphi : G \to H$ will be called an R-**homomorphism** if $\varphi(g^\alpha) = \varphi(g)^\alpha$, $g \in G$, $\alpha \in R$.

For basic definitions in category $\mathfrak{M}_R$ and results concerning exponential MR-group see [2].

Let R be an arbitrary associative ring unit. Then **the class $\mathfrak{M}_R$ ($\mathfrak{L}_R$) is a category in which morphisms are R-homomorphism of groups**.

Below we prove that the classes $\mathfrak{L}_R$ and $\mathfrak{M}_R$ are closed un under direct and Cartesian products and under direct and inverse limits.

Let $G_i \in \mathfrak{L}_R$, $i \in I$. We denote by $\overline{\prod} G_i$ and $\prod G_i$, respectively, the Cartesian and direct products of the groups G_i. Let $g \in \overline{\prod} G_i$, $g = (\dots, g_i, \dots)$, and $\alpha \in R$. We define an action of R on G by the coordinate-wise rule $g^\alpha = (\dots, g_i^\alpha, \dots)$. It can be immediately proved that if all groups G_i satisfy one of axioms (i)–(iv), then the groups $\overline{\prod} G_i$ and $\prod G_i$, also satisfy this axiom. Thus, we have proved

Proposition 1. *The classes $\mathfrak{L}_R$ and $\mathfrak{M}_R$ are closed with respect to direct and Cartesian product.*

If in the standard definitions of direct and inverse spectrums one considers only R-homomorphisms then it is not difficult to prove

Proposition 2. *The classes $\mathfrak{L}_R$ and $\mathfrak{M}_R$ are closed with respect to direct and inverse limits.*

It is proved in [11] that in Abelian group category the operations of direct product of groups, of direct and inverse limits have a universal property. The corresponding actions in exponential MR-group category have analogous properties.

Definition 4 ([2])**.** Let G be an MR-group and let $\mu : R \to S$ be a homomorphism of rings. Then a S-group $G^{S,\mu}$ is called the **tensor S-completion** of an MR-group G, if $G^{S,\mu}$ satisfies the following universal property:

(1) there exists and R-homomorphism $\lambda : G \to G^{S,\mu}$ such that $\lambda(G)$ S-generators $G_{s,\mu}$, i.e. $\langle \lambda(G) \rangle = G^{S,\mu}$;

(2) for any S-group H and any R-homomorphism $\varphi : G \to H$ coordinated with μ (that $\varphi(g^\alpha) = \varphi(g)^{\mu(\alpha)}$) there exists a S-homomorphism $\psi : G^{S,\mu} \to H$, rendering the

following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\lambda} & G^{S,\mu} \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle \exists\psi} & \\ H & & \end{array} \qquad (\varphi = \lambda\psi).$$

Note that if G is an Abelian MR-group, then $G^{S,\mu} \cong G \bigotimes\limits_{R} S$ is a tensor product of an R-module G by a ring S. In [2] it is proved that for any MR-group G and any homomorphism $\mu : R \to S$ the tensor completion $G^{S,\mu}$ always exists and it is unique to written an isomorphism.

2 Commutation of the functor of tensor completion with basic group operations. Let $G_i \in \mathfrak{M}_R$, $i \in I$.

Theorem 1. *If* $G = \prod\limits_i G_i$, *then* $G^S = \prod\limits_i G_i^S$.

Theorem 2. *If* $G_* = \varinjlim\limits_i G_i$, *then* $G_*^S = \varinjlim\limits_i G_i^S$.

Remark 1. Let us give an example showing that the operation of Cartesian product is not commutable with the operation of tensor completion. Denote $\lambda : \overline{\prod\limits_i} G_i \to \overline{\prod\limits_i} G_i^S$. Then by the universal property of tensor completion, we have a S-homomorphism $\lambda^s : \big(\overline{\prod\limits_i} G_i\big)^S \to \overline{\prod\limits_i} G_i^S$, which is not an isomorphism in the general case. Such an example exists in the theory of Abelian groups. Let us take the field of rational numbers $\mathbb{Q}$ as the ring R a cyclic group of order n as G_n. Let $G_n = \langle a_n \rangle$, $n \in \mathbb{N}$. Then $G_n^{\mathbb{Q}} = G_n \otimes \mathbb{Q} = 0$. Therefore, $\overline{\prod} G_n^{\mathbb{Q}} = 0$. At the same time there exist elements of infinite order in the group $\overline{\prod\limits_n} G_n$, therefore the group $\big(\overline{\prod\limits_n} G_n\big)^{\mathbb{Q}} = \overline{\prod\limits_n} G_n \otimes \mathbb{Q}$ is nonzero.

Remark 2. Let G^* be the limits group of inverse spectrum $\mathbb{G}^* = \{G_i,\ i \in I,\ \pi_i^j\}$. By the use of universal property of inverse limit we shall construct a S-homomorphism $\sigma : (G^*)^S \to \varprojlim\limits_i G_i^S$. To do this we denote by $\pi_i : G^* \to G_i$ the projection of the limit group onto the component with index i. Then $\pi_i^s : (G^*)^S \to G_i^S$ is the corresponding homomorphism of a tensor completion. Let $\mu_i : \varprojlim\limits_i G_i^S \to G_i^S$ be a natural project. Then according to universal property of inverse limits, there exists a homomorphism $\sigma : (G^*)^S \to \varprojlim\limits_i G_i^S$, making the diagram

$$\begin{array}{ccc} (G^*)^S & \xrightarrow{\quad\sigma\quad} & \varprojlim\limits_i G_i^S \\ & {\scriptstyle\pi_i^s}\searrow \quad \swarrow{\scriptstyle\mu_i} & \\ & G_i^S & \end{array}$$

commutative.

We shall demonstrate with an example that this homomorphism σ is not isomorphism in the general case. Let us consider G_n, $n \in \mathbb{N}$, $G_n = \langle a_n \rangle$, where a_n is an element of order p^n, p is a prime number. Then it is known that $\varprojlim_n G_n \cong \mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty}$ is the additive group of integral p-adic numbers, $\mathbb{Z}_{p^\infty}^{\mathbb{Q}} = \mathbb{Z}_{p^\infty} \otimes \mathbb{Q}$ is a vector space over $\mathbb{Q}$ of continual cardinality. At the same time $\varprojlim_n G_n^{\mathbb{Q}} = \varprojlim_n (G_n \otimes \mathbb{Q}) = \varprojlim o = 0$.

REFERENCES

1. Lyndon, R.C. Groups with parametric exponents. *Trans. Amer. Math. Soc.*, **96** (1960), 518-533.
2. Myasnikov, A.G., Remeslennikov, V.N. Degree groups. I. Foundations of the theory and tensor completions (Russian). *Sibirsk. Mat. Zh.*, **35**, 5 (1994), 1106-1118; translation in *Siberian Math. J.*, **35**, 5 (1994), 986-996.
3. Amaglobeli, M. Precise exponential MR-groups. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **28** (2014), 1-4.
4. Myasnikov, A.G., Remeslennikov, V.N. Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups. *Inter. J. Algebra Comput.*, **6**, 6 (1996), 687-711.
5. Baumslag, G., Myasnikov, A., Remeslennikov, V. Discriminating completions of hyperbolic groups. Dedicated to John Stallings on the occasion of his 65th birthday. *Geom. Dedicata*, **92** (2002), 115-143.
6. Amaglobeli, M.G., Remeslennikov, V.N. Free nilpotent R-groups of class 2 (Russian). *Dokl. Akad. Nauk*, **443**, 4 (2012), 410-413; translation in *Dokl. Math.*, **85**, 2 (2012), 236-239.
7. Amaglobeli, M.G., Remeslennikov, V.N. Extension of a centralizer in nilpotent groups (Russian). *Sibirsk. Mat. Zh.*, **54**, 1 (2013), 8-19.
8. Amaglobeli, M., Remeslennikov, V. Algorithmic problems for class-2 nilpotents MR-groups. *Georgian Math. J.*, **22**, 4 (2015), 441-449.
9. Amaglobeli, M.G., Remeslennikov, V.N. Fundamentals of the theory of varieties of nilpotent MR-groups (Russian). *Sib. Mat. Zh.*, **57**, 6 (2016), 1197-1207; translation in *Sib. Math. J.*, **57**, 6 (2016), 935-942.
10. Baumslag, G. Free Abelian X-groups. *Illinois J. Math.*, **30**, 2 (1986), 235-245.
11. Fuchs, L. Infinite Abelian Groups. Vol. I. Pure and Applied Mathematics, Vol. 36 *Academic Press, New York–London*, 1970.

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